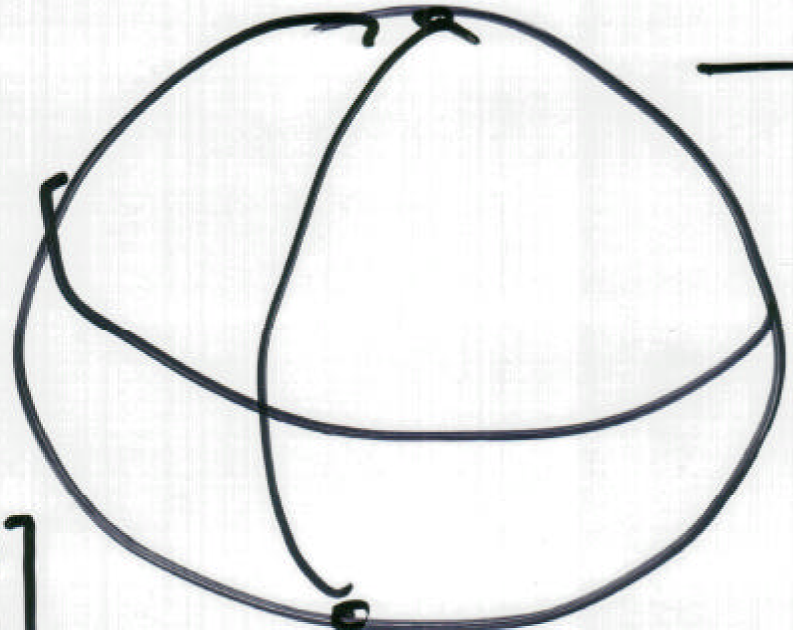
ROTO-TRASLAZIONI IN 3D

- CAP 2 TESTO
- APPENDICE
- APPUNTI MATRICI & VETTORI
- LEZIONI 2ª SETTIMANA 2004

GEOMETRIA
- PIANE
- 3D
- ....

→ EUCLIDEA



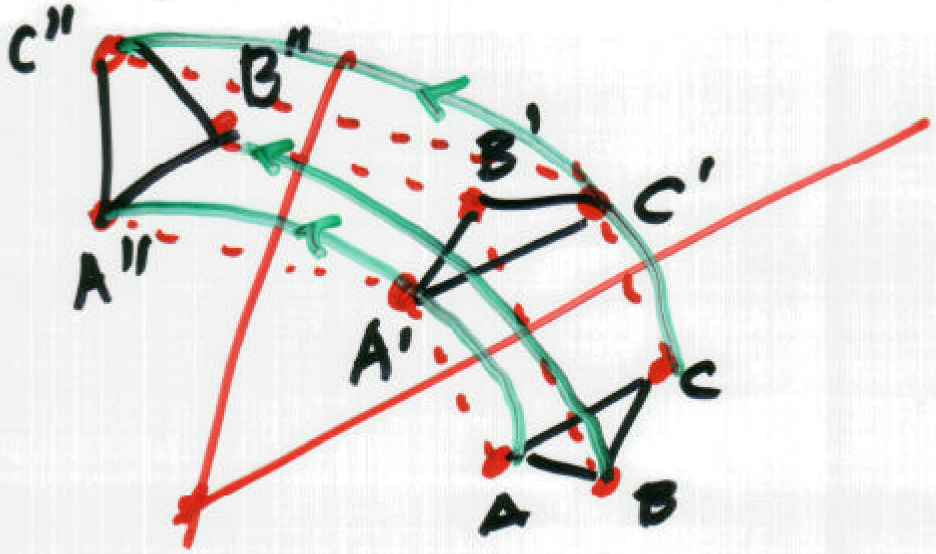
$P_1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ $\quad$ $P_2 \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$$d(P_1 P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

TRASFORMAZIONI ISOMETRICHE

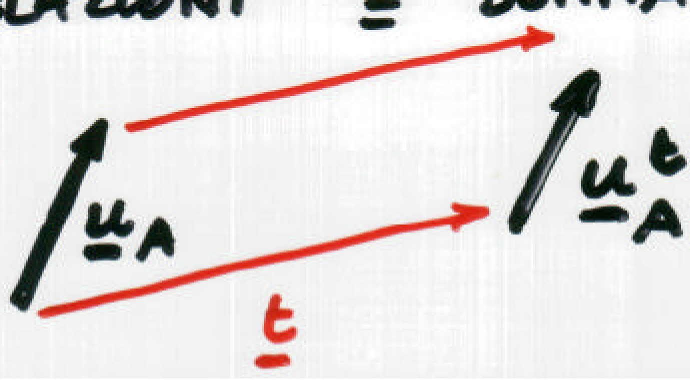
CONSERVAZIONE DELLE DISTANZE

- RIFLESSIONI → IN ROBOTICA NON SI USA
- ROTAZIONI
- TRASLAZIONI



ROTAZIONI + TRASLAZIONI

TRASLAZIONI ≡ SOMMA VETTORIALE

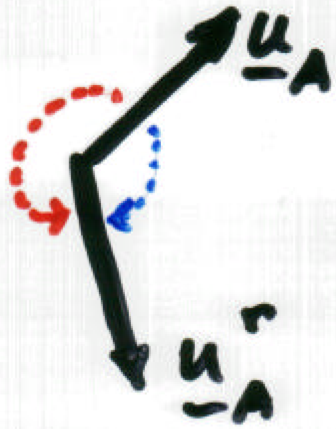


$$\underline{u}_A^t = \underline{u}_A + \underline{t}$$

$$\underline{u}_A = \underline{u}_A^t - t$$

ROTAZIONI

$$\underline{u}_A \rightsquigarrow ? \ \underline{u}_A^r$$



$$\underline{u}_A^r = R\,\underline{u}_A$$

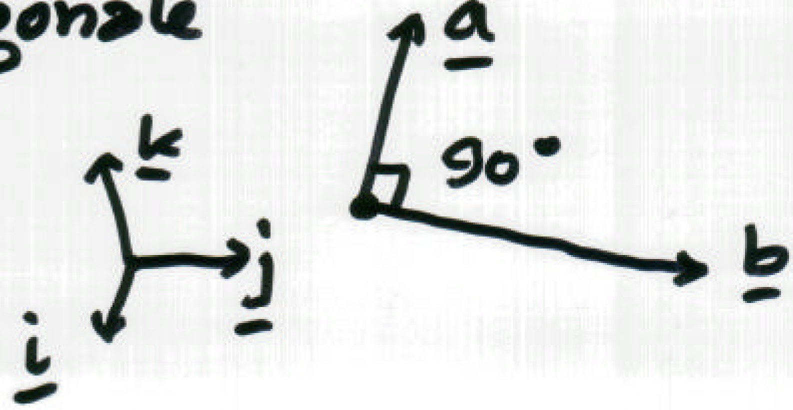
OPERATORE : PRODOTTO DI MATRICE

$$\underline{u}_A = R^{-1}\,\underline{u}_A^r$$

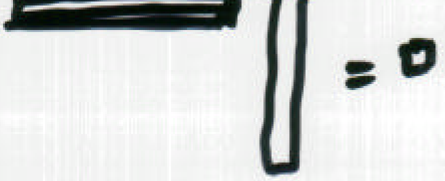
$$R^{-1} = R^T$$

R 3×3 di rotazione

- ORTO NORMALE righe/colonne ortogonali hanno norma 1 ortogonale



$\underline{a}^T \underline{b} = 0$ prodotto scalare è nullo



$\underline{a} \cdot \underline{b} = \underline{a}^T \underline{b}$

$\underline{a} \times \underline{b}$ ←

$\underline{a} \wedge \underline{b}$

$$R = \begin{bmatrix} | & | & | \\ \underline{r}_1 & \underline{r}_2 & \underline{r}_3 \\ | & | & | \end{bmatrix}$$

$\underline{r}_i^T \underline{r}_i = 1$

$\underline{r}_i^T \underline{r}_j = 0$

$$R = \begin{bmatrix} - c_1 - \\ - c_2 - \\ - c_3 - \end{bmatrix}$$

$\underline{c}_i^T \underline{c}_i = 1$

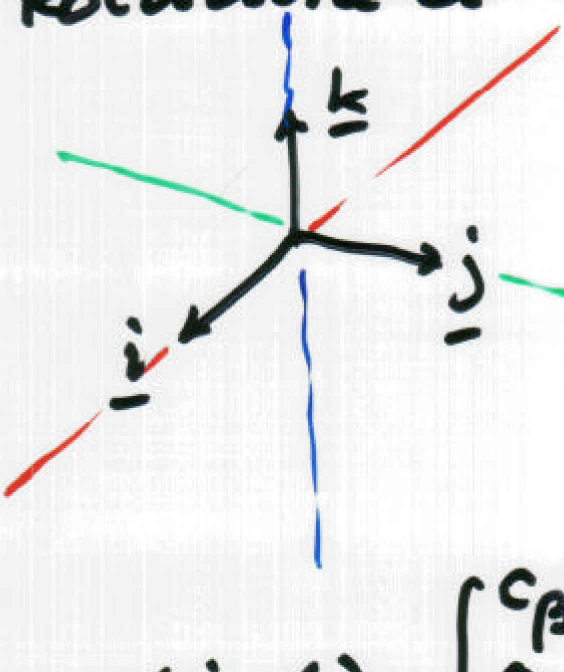
$\underline{c}_i^T \underline{c}_j = 0$

- $R^{-1} = R^T \qquad RR^T = R^TR = I$

- $\det(R) = +1$

Riflessioni $\bar{R}$ $\quad \det(\bar{R}) = -1$

Rotazione di α intorno a $\underline{i}$ (α)



$$Rot(\underline{i}, \alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\alpha & -S_\alpha \\ 0 & S_\alpha & C_\alpha \end{bmatrix}$$

$$Rot(\underline{j}, \beta) = \begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix}$$

$$Rot(\underline{k}, \gamma) = \begin{bmatrix} C_\gamma & -S_\gamma & 0 \\ S_\gamma & C_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$Rot(\underline{u}, \theta) \Rightarrow$ formula generale sul testo

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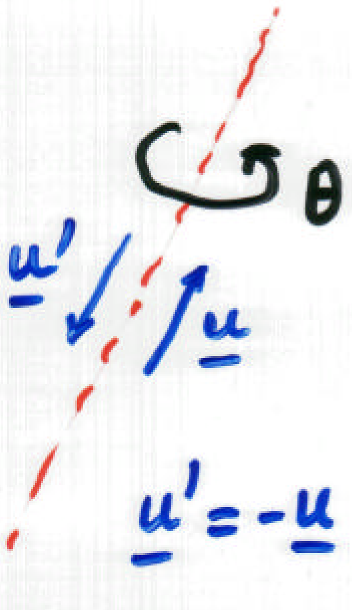
$Rot(\underline{u}, \theta)$ (2.35)

$$Rot(\underline{u}, -\theta) = R(\underline{u}, 2\pi - \theta) = R(-\underline{u}, \theta)$$



$\underline{u}' = -\underline{u}$

VERSORE = VETTORE $\|\cdot\| = 1$

$\|\underline{u}\| = 1$

$\underline{u}^T \underline{u} = 1$

$$Rot(\underline{u}, \theta) \cdot Rot(\underline{u}', \phi) \cdot Rot(\underline{u}'', \psi)$$

$$R_1 R_2 R_3 \dots R_n$$

$$R_1 R_2 \neq R_2 R_1$$

IN GENERALE

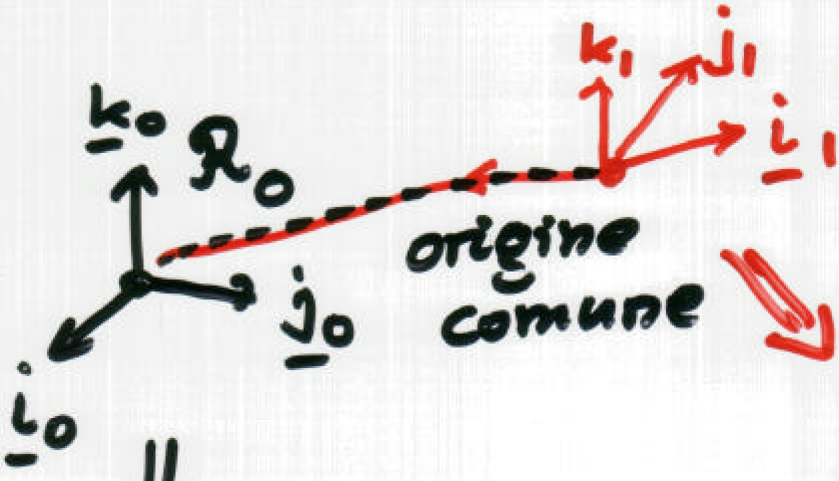
$$\mathrm{Rot}(\underline{u},\theta)\cdot \mathrm{Rot}(\underline{u},\phi)\cdot \dots \mathrm{Rot}(\underline{u},\psi)$$

$$= \mathrm{Rot}(\underline{u},(\theta+\phi+\dots+\psi))$$

$$\mathrm{Rot}(\underline{u},\phi)\,\mathrm{Rot}(\underline{u},\psi)\dots \mathrm{Rot}(\underline{u},\theta)$$

SOLO ROT. PIANE

$$R_1 R_2 R_3 R_4 = R_2 R_3 R_4 R_1$$



$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathcal{R}_1}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathcal{R}_0}$$

$\mathcal{R}_0$ in $\mathcal{R}_1$

oppure

$\mathcal{R}_1$ in $\mathcal{R}_0$

se $\mathcal{R}_0$ è quello "fisso"

$\mathcal{R}_1$ in $\mathcal{R}_0$ cos'è?

$$\begin{bmatrix} \Big| & \Big| & \Big| \end{bmatrix}$$

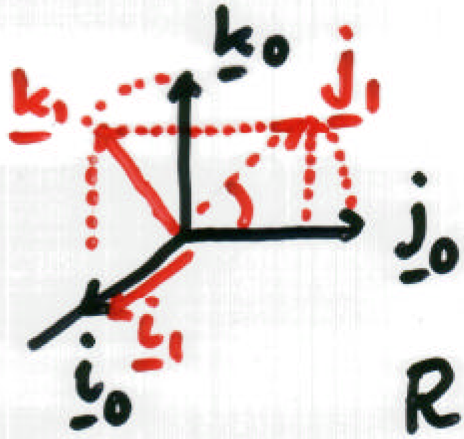
$\underline{k}_1$ in $\mathcal{R}_0$

$\underline{j}_1$ in $\mathcal{R}_0$

$\underline{i}_1$ " $\mathcal{R}_0$

ESEMPIO

$\mathrm{Rot}(\underline{i}_0, 45°)$



$$
\begin{array}{cc}
 & \begin{array}{ccc} \underline{i}_1 & \underline{j}_1 & \underline{k}_1 \\ \downarrow & \downarrow & \downarrow \end{array} \\
R = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}
\end{array}
$$

R rappresenta $\mathcal{R}_1$ in $\mathcal{R}_0 \rightarrow R_1^0$ SI

${}^0R_1$ NO

R_{01} NO

R_{10} NO

R_1^0 che rappresenta $\mathcal{R}_1$ in $\mathcal{R}_0$
è anche la trasformazione "fisica"
che porta $\mathcal{R}_0$ a sovrapporsi a $\mathcal{R}_1$

— o —

$$R = I R = R I$$

$$\boxed{\begin{array}{l}\text{PRE-FISSO} \\ \text{POST-MOBILE}\end{array}}$$

ROTO-TRASLAZIONI

$\downarrow$

R_j^i

$i \leftarrow$ SISTEMA "FISSO" $\mathcal{R}_i$

$j \leftarrow$ " "MOBILE" $\mathcal{R}_j$

matrice 3×3 3D

•) TRASFORMA UN VETTORE

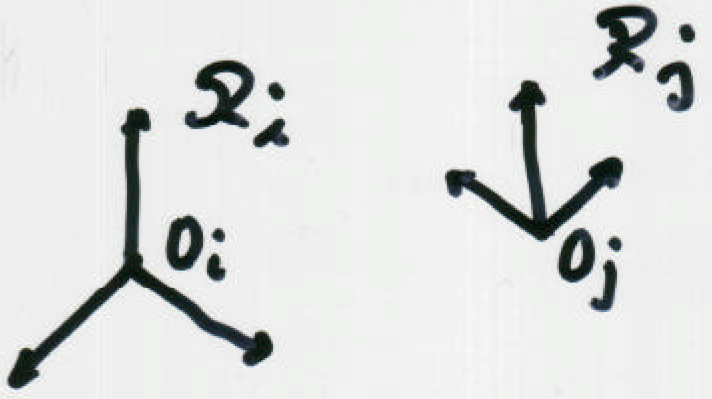
$$[\underline{x}]_{\mathcal{R}_j} \longrightarrow [\underline{x}]_{\mathcal{R}_i} = R_j^i [\underline{x}]_{\mathcal{R}_j}$$

•) R_j^i contiene, per colonne

$$\begin{bmatrix} | & | & | \end{bmatrix}$$ i versori di $\mathcal{R}_j$

rappresentati in $\mathcal{R}_i$

•) R_j^i descrive la rotazione che porta $\mathcal{R}_i$ a sovrapporsi a $\mathcal{R}_j$



$O_i \equiv O_j$

$$R_j^i \longrightarrow \left[R_j^i\right]^{-1} \equiv \left[R_j^i\right]^T = R_i^j$$

$$Rot(\underline{i}, 45) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = R_1$$

$$Rot(\underline{i}, -45°) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = R_1^T$$

una è l'inverso dell'altra

COME COMPONGO n ROTAZIONI

1) individuo la rotazione
2) individuo se la rotazione avviene rispetto agli assi "fissi" oppure "mobili"

3) PRE MOLTIPLICO SE "FISSO"
POST MOLTIPLICO SE "MOBILI"

ESEMPIO

ANGOLI DI EULERO

1) $Rot(\underline{k}, \phi)$ assi mobili
2) $Rot(\underline{i}, \theta)$ " "
3) $Rot(\underline{k}, \psi)$ " "

$$Rot(\underline{k}, \phi) = R_1 = \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Rot(\underline{i}, \theta) = R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}$$

$$Rot(\underline{k}, \psi) = R_3 = \begin{bmatrix} C_\psi & -S_\psi & 0 \\ S_\psi & C_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(I) $R_1 R_2 R_3$

$$Rot_{EULERO} = R_1 R_2 R_3 \quad (2.73)$$

$R_a \; R_b \; R_c$

1 2 3

3 1 2

2 1 3

3 2 1

R_c

$R_b R_c$

$R_a(R_b R_c)$

(R_b)

$(R_b R_c)$

$R_a(R_b R_c)$

R_b

$R_a R_b$

$(R_a R_b) R_c$

$R_c' R_c'' R_c''$

$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R_b = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad R_a R_b R_c$$

ROTO + TRASLAZIONE

$\Downarrow$

VETTORI OMOGENEI
(HOMOGENEOUS VECTORS)

$$\tilde{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \longleftarrow \text{STRUTTURALE}$$

$$x \rightarrow \tilde{x} = \begin{bmatrix} x \\ \cdots \\ 1 \end{bmatrix}$$

$$\tilde{x} + \tilde{y} = \, ? = \begin{bmatrix} x+y \\ \cdots\cdots \\ 2 \end{bmatrix} \; !!! \quad \underline{\text{NO}} \qquad = \begin{bmatrix} x+y \\ \cdots \\ 1 \end{bmatrix}$$

SI

$\underline{\text{ROTO}}$ - $\underline{\text{TRASLAZIONI}}$ = MATRICI 4×4

$$T = \left[\begin{array}{c|c} R_{3\times3} & t_{3\times1} \\ \hline 0 \; 0 \; 0 & 1 \end{array}\right]$$

COME SI ROTOTRASLA UN VETTORE $\underline{x}$?

1) $\underline{x} \rightarrow \underline{\tilde{x}}$

2) $\underline{\tilde{x}}' = T \underline{\tilde{x}}$

3) $\underline{\tilde{x}}' \rightarrow \underline{x}'$

$$T = T_1 T_2 T_3 T_4 T_5 \dots$$

PRE - FISSO
POST - MOBILE

$$T \stackrel{?}{=} Rot(\underline{i}, \alpha) \qquad T = \begin{bmatrix} R_{\underline{i},\alpha} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}$$

$$T \stackrel{?}{=} Rot(\underline{j}, \beta) = T = \begin{bmatrix} R_{\underline{j},\beta} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}$$

Rotazione semplice R

$$T = \begin{bmatrix} R & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}$$

Traslazione semplice $\underline{t}$

$$T = \begin{bmatrix} I & \underline{t} \end{bmatrix}$$

$$T_a = \begin{bmatrix} R & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \qquad T_b = \begin{bmatrix} I & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix}$$

$T_a T_b$

$T_b T_a$

I) $T_a T_b$
- Rot + trasl mobile
- trasl + Rot fisso

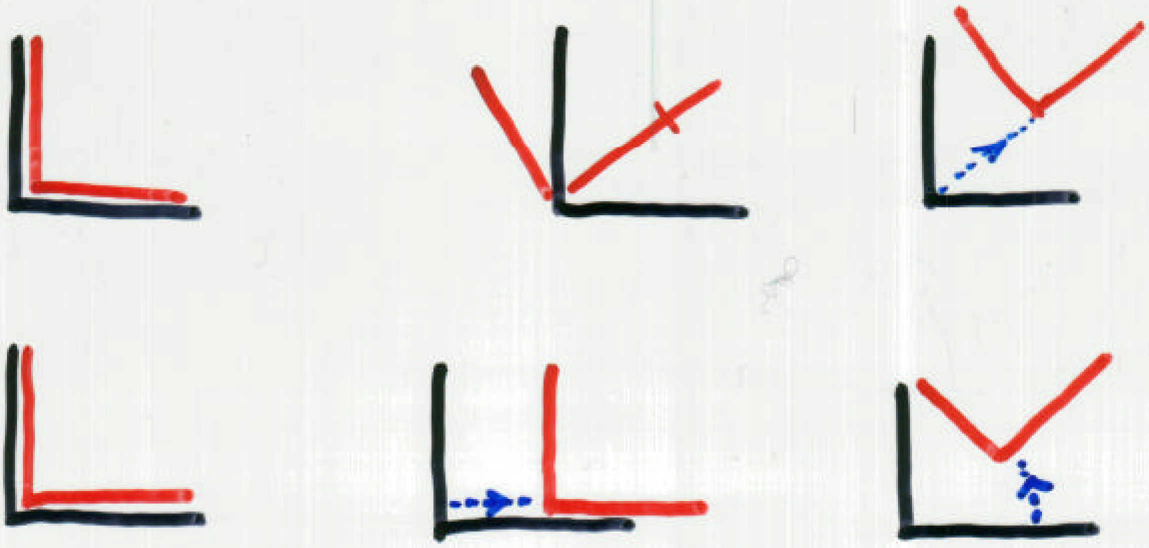
$$\begin{bmatrix} R & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}\begin{bmatrix} I & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R & Rt \\ 0^T & 1 \end{bmatrix}$$

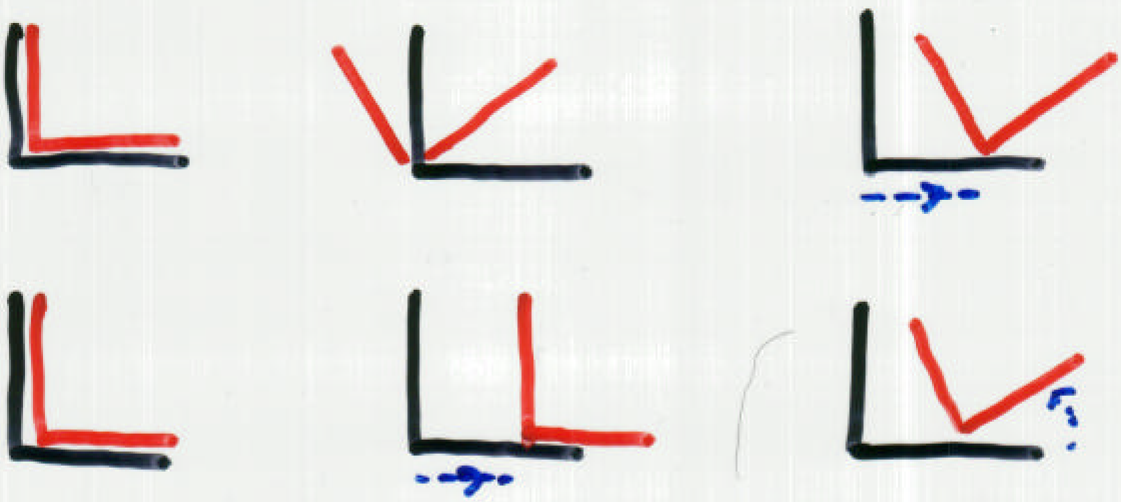
$$\underline{0}\,\underline{0}^T = \begin{bmatrix} \\ \\ \\ \end{bmatrix}\begin{bmatrix} & & \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a^T b$

$a\, b^T$

II) $T_b T_a$ $\quad \begin{bmatrix} I & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix}\begin{bmatrix} R & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$
- Rot + trasl fisso
- trasl + Rot mobile





R matrice di rotazione $\underbrace{3 \times 3}_{\text{9 elementi}}$

9 elementi, ma solo 3 parametri sono essenziali

data R come estrarre i 3 elementi ?

esistono altri modi di rappresentare l'assetto?

— o —

dare i 3 elementi direttamente

↓ angoli

1) R 9 elem
2) ANGOLI < DI EULERO / RPY
   3 angoli
3) ASSE - ANGOLO $\underline{u}$ θ 1 per
   $\|\underline{u}\| = 1$ 2 per

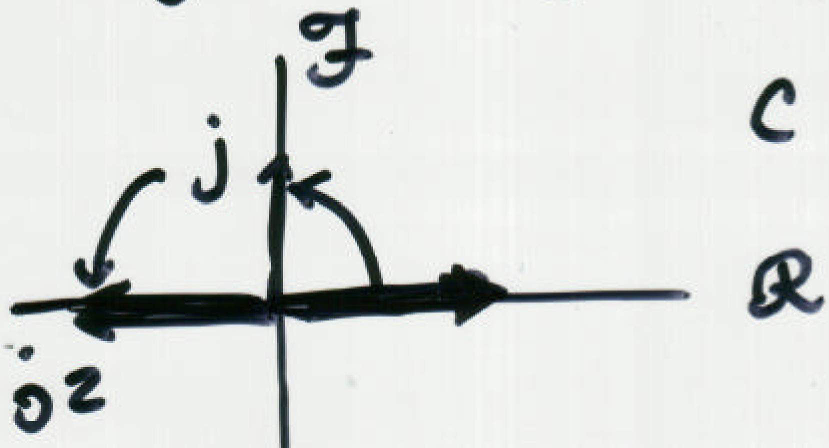
4) QUATERNIONI (UNITARI)

5) VETTORI DI ROTAZIONE DI RODRIGUES

6) MATRICI DI DIRAC 2x2

QUATERNIONI

$a + jb \qquad j^2 = -1$



$h = h_0 + i\,h_1 + j\,h_2 + k\,h_3$

$i^2 = j^2 = k^2 = -1$

$ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j$

$ijk = 1$

$$h = \underbrace{h_0}_{\text{PARTE REALE}} + \underbrace{i h_1 + j h_2 + k h_3}_{\text{PARTE VETTORIALE}}$$

$h = (h_0, 0, 0, 0)$ reali

$h = (h_0, h_1, 0, 0)$ complessi

$h = (0, h_1, h_2, h_3)$ vettori

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norma di quaternione

$$\|h\| = \sqrt{h_0^2 + h_1^2 + h_2^2 + h_3^2}$$

unitario $\|h\| = 1$

$$\underline{h} = \sin\frac{\theta}{2} + i\, u_1 \cos\frac{\theta}{2} + j\, u_2 \cos\frac{\theta}{2} + k\, u_3 \cos\frac{\theta}{2}$$

$\|h\| = 1$

n rotazioni $R_1 \cdots R_n$

$$\begin{array}{ccc} R_1 & \cdots & R_n \\ \downarrow & & \downarrow \\ h_1 & & h_n \end{array}$$

$$R_1 \cdots R_n = R \longrightarrow h$$

$$h = h_1 h_2 \cdots h_n$$

— o —

$ijk = -1$

ϕ, θ, ψ Eulero

$$R(\underline{k}, \phi)\, R(\underline{i}, \theta)\, R(\underline{k}, \psi)$$

Roll - Pitch - Yaw

$\theta_x, \theta_y, \theta_z$

$$R(\underline{k}, \theta_z)\, R(\underline{j}, \theta_y)\, R(\underline{i}, \theta_x)$$

R

asse - angolo

angoli di Eulero

angoli di RPY

quaternione

$$T = \begin{bmatrix} R & t \\ \dots & . \end{bmatrix}$$