SET MEMBERSHIP IDENTIFICATION THEORY

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III Level Course 01LCPIU

“Experimental modeling: model building from experimental data”

IBC estimation problem

- Given:
  - λ: *unknown* problem element, e.g. $\begin{cases}\text{dynamical system}\\ \text{time function}\end{cases}$
  - S: *known* "solution" operator, e.g. $\begin{cases}\text{parameters of the dynamical system}\\ \text{values of the time function}\end{cases}$
- Available information:
  - "a priori" information

    on the problem element: $\lambda \in K \subseteq \Lambda$

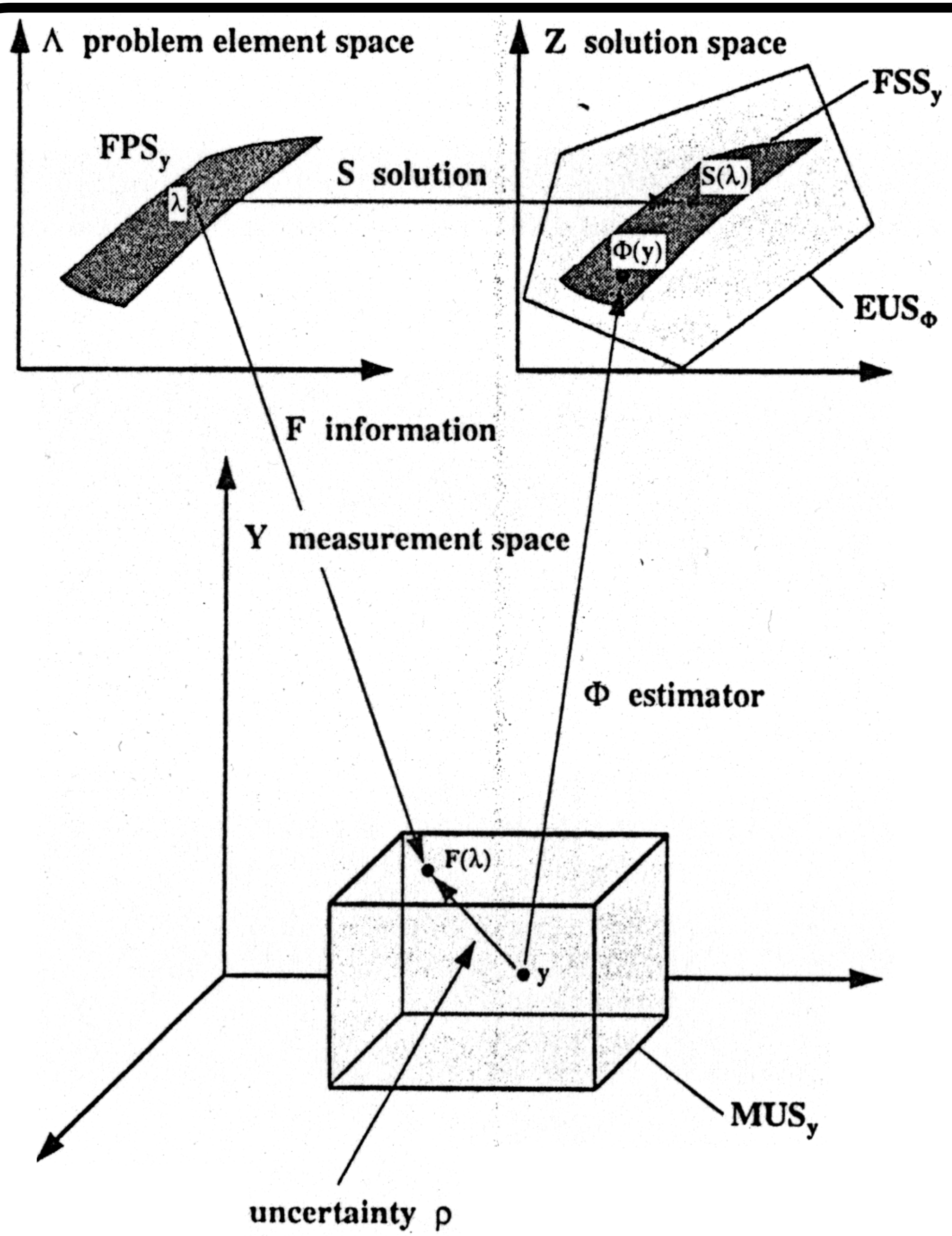
    on the measurement noise: $e^N \in \mathcal{B}_e = \left\{ e^N \in \Re^N : \left\| e^N \right\| \leq \varepsilon \right\}$
  - measurement information ("a posteriori" information)

$$\underbrace{y^N}_{\textit{known}\text{ data}} = \underbrace{F(\lambda)}_{\textit{known}\text{ function}} + \underbrace{e^N}_{\textit{unknown}\text{ noise}} \quad , \quad F\text{: \textit{known} ``information'' operator}$$

- Estimation problem:
  1. find an estimation algorithm ϕ that well approximates $S(\lambda)$

$$\phi\left(y^N\right) = \hat{z} \approx S(\lambda)$$

  2. evaluate the approximation quality



λ	:	problem element $\in K$
$S(\lambda)$	:	function of λ to be estimated
$F(\lambda)$	:	"noise-free" measurement information
y^N	:	actual measurement information
e^N	:	measurement noise $\in \mathcal{B}_e$
$\phi\left(y^N\right)$	:	approximation of $S(\lambda)$

Example: parameter estimation for an $\mathbf{ARX}\,(\mathbf{n_a}, \mathbf{n_b})$ model

$$y_j = \sum_{i=1}^{n_a} a_i y_{j-i} + \sum_{i=1}^{n_b} b_i u_{j-i} + e_j, \quad j = 1, \ldots, N'$$

- $\Lambda : (n_a + n_b)$-dimensional space with elements

$$\lambda = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n_a} & b_1 & b_2 & \cdots & b_{n_b} \end{bmatrix}^T \in \Re^r, \quad r = n_a + n_b$$

- $Z : (n_a + n_b)$-dimensional space with elements $z = \lambda \Rightarrow Z \equiv \Lambda$
- $S = I =$ identity operator
- $Y : (N' - n_a)$-dimensional space with elements

$$y = \begin{bmatrix} y_{n_a+1} & y_{n_a+2} & \cdots & y_{N'} \end{bmatrix}^T \in \Re^N, \quad N = N' - n_a$$

- $F(\lambda) = L \cdot \lambda$, with $L = \begin{bmatrix} y_{n_a} & \cdots & y_1 & u_{n_a} & \cdots & u_{n_a+1-n_b} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{N'-1} & \cdots & y_{N'-n_a} & u_{N'-1} & \cdots & u_{N'-n_b} \end{bmatrix} \in \Re^{N \times r}$

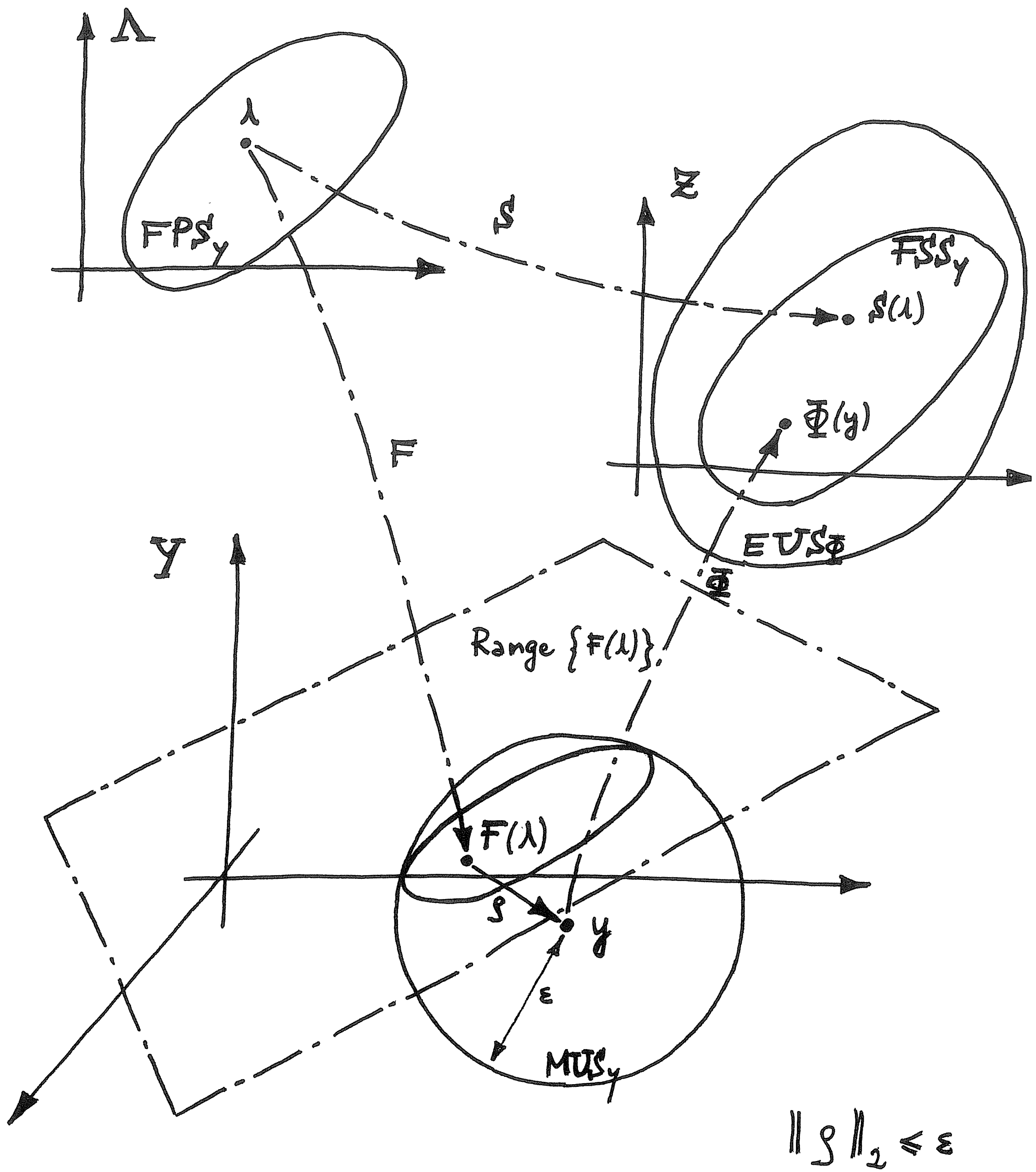
  $\Rightarrow F(\lambda)$ is *linear* in λ

Main sets of interest

- $MUS_y = \{\tilde{y} \in Y : ||\tilde{y} - y|| \leq \varepsilon\}$: **Measurement Uncertainty Set**
- $EUS_\phi = \phi\left(MUS_y\right)$: **Estimate Uncertainty Set**
  - EUS_ϕ depends obviously on the estimation algorithm ϕ
  - EUS_ϕ provides all the estimates of $S\left(\lambda\right)$ obtained by considering all the possible measurements consistent with the overall information
- $FPS_y = \{\lambda \in K : ||y - F\left(\lambda\right)|| \leq \varepsilon\}$: **Feasible Problem Element Set**
- $FSS_y = S(FPS_y)$: **Feasible Solution Set**
  - FSS_y depends on the problem formulation only
  - FSS_y provides all the values of $S\left(\lambda\right)$ consistent with the overall information
  - In case of parametric estimation, $S\left(\lambda\right) = \lambda$

$$\Downarrow$$

$$FPS_y \equiv FSS_y$$: **Feasible Parameter Set**



$\lambda \in K \subseteq \Lambda$: *unknown* problem element

S : *known* solution operator

F : *known* information operator

y : *known* measurement information

$e \in \mathcal{B}_e$: *unknown* measurement noise

Measurement Uncertainty Set:

$$MUS_y = \{\tilde{y} \in Y : ||\tilde{y} - y|| \leq \varepsilon\}$$

Estimate Uncertainty Set:

$$EUS_\phi = \phi(MUS_y)$$

Feasible Problem Element Set:

$$FPS_y = \{\lambda \in K : ||y - F(\lambda)|| \leq \varepsilon\}$$

Feasible Solution Set:

$$FSS_y = S(FPS_y)$$

Errors and optimality concepts

- $E_y(\phi) = \sup_{\lambda \in FPS_y} \| S(\lambda) - \phi(y) \| = \sup_{S(\lambda) \in FSS_y} \| S(\lambda) - \phi(y) \|$: **local error**
- $E(\phi) = \sup_{y \in Y} E_y(\phi)$: **global error**
- An estimation algorithm that minimizes either of these errors is called, respectively:
  - locally optimal
  - globally optimal
- Note that: local optimality $\overset{\Longrightarrow}{\not\Longleftarrow}$ global optimality

Classes of estimators

1. **Central estimators**

$$\phi^c(y) = \text{ Chebicheff center of } FSS_y := c(FSS_y) \in Z$$

is the center of the minimal ball containing FSS_y

$$\text{rad}(FSS_y) := \sup_{z \in FSS_y} \|c(FSS_y) - z\| = \inf_{\tilde{z} \in Z} \sup_{z \in FSS_y} \|\tilde{z} - z\|$$

is the radius of the minimal ball containing FSS_y

2) CORRECT estimators

$$\Phi(F(\lambda)) = S(\lambda) \qquad \forall \lambda \in \Lambda$$

give exact solution if applied to exact info

- projection estimators

$$\Phi^{P}(y) = S(\lambda_y), \qquad \lambda_y \in \Lambda \text{ s.t.}$$

$$\| y - F(\lambda_y) \| = \inf_{\lambda \in \Lambda} \| y - F(\lambda) \|$$

- $\exists \neq$ projectors estimators according to the norm used in Y:
  - ℓ_2 $\left[\|y\|_2 = \sqrt{\sum_{i=1}^{m} |y_i|^2} \right]$ → Least Squares Algorithms Φ^{LS}
  - ℓ_1 $\left[\|y\|_1 = \sum_{i=1}^{m} |y_i| \right]$ → Least Absolute Values Algo
  - ℓ_∞ $\left[\|y\|_\infty = \max_{i=1,\dots,m} |y_i| \right]$ → Least Maximum Values Algo [Minimax Algo]

Estimator Properties for LINEAR Problems $\left[\begin{array}{l} S(\lambda) \rightarrow S_N \cdot \lambda \\ F(\lambda) \rightarrow F_N \cdot \lambda \end{array}\right]$

- CORRECT estimators

$$FSS_y \subseteq EUS_{\Phi} \quad \forall y \in Y$$

[Milanese - Belforte, '82]

- PROJECTION estimators
  - "Almost" local optimality

$$E_y(\Phi^p) \leq 2\, rad\,(FSS_y) \leq 2\, E_y(\Phi),$$

$$\forall y \in Y, \quad \forall \Phi$$

[Kacewicz - Milanese - Tempo - Vicino, '86]

- LEAST SQUARES estimators
  - Linearity (by construction)

$$\Phi^{LS}(y) = (F_N^T F_N)^{-1} F_N^T y$$

  - Correctness (by definition) =>

$$FSS_y \subseteq EUS_{\Phi^{LS}}, \quad \forall y \in Y$$

  - if Y is ℓ_2 - NORMED
    - Centrality: $\Phi^{LS}(y) = c(FSS_y)$
    - Local Optimality: $E_y(\Phi^{LS}) \leq E_y(\Phi)$,

$$\forall y \in Y, \ \forall \Phi$$

[Kacewicz - Milanese - Tempo - Vicino, '86]

Basic IBC-SM results

CONVERGENCE CONCEPTS

- ϕ is *convergent* if:

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} E_y(\phi) = 0, \quad \forall x \in K \quad \forall y$$

- ϕ is *globally convergent* if:

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} E(\phi) = 0$$

global convergence $\overset{\not\Leftarrow}{\Rightarrow}$ convergence