System identification, Estimation and Filtering

Consider the following LTI system:

$$\begin{aligned} x(t+1) &= Ax(t) + v_1(t) \\ y(t) &= Cx(t) + v_2(t) \end{aligned} \tag{1}$$

where

$$A = \begin{bmatrix} 0.96 & 0.5 & 0.27 & 0.28 \\ -0.125 & 0.96 & -0.08 & -0.07 \\ 0 & 0 & 0.85 & 0.97 \\ 0 & 0 & 0 & 0.99 \end{bmatrix}, \quad C = [0 \quad 2 \quad 0 \quad 0]$$

$v_1(t)$ is a white noise with zero mean value and variance $V_1 = B_v B_v^T$, $B_v = \sqrt{15}[0.5 \ 0 \ 0 \ 1]^T$, and $v_2(t)$ is a white noise with zero mean value and variance $V_2 = 2000$. The noises $v_1(t)$ and $v_2(t)$ are uncorrelated. Assume that the initial state $x(1)$ is a random vector with zero mean value and variance $P_1 = E\left[x(1)x(1)^T\right] = 0.5I_4$.

Problem: Design the following predictors/filters:
- Dynamic Kalman 1-step predictor.
- Dynamic Kalman filter.
- Steady-state Kalman 1-step predictor.
- Steady-state Kalman filter.
- (Optional) Dynamic Kalman 1-step predictor in predictor/corrector form.

Compare the estimates provided by these predictors/filters by means of graphical representations and evaluation of the Root Mean Square Error ($RMSE$).

Main steps:

(1) Create a Matlab program for the simulation of the system (1). Use a *for* loop in order to implement the simulation. In the program, the noise $v_1(t)$ can be generated as *v1(:,t)=Bv.*randn(4,1);* the noise $v_2(t)$ can be generated as *v2(t)=sqrt(V2)*randn;*

(2) Simulate the system for $t = 1, 2, ..., N$, $N = 2000$ starting from random initial conditions: $x(1) = \sqrt{0.5}randn(4, 1)$. Plot the four state signals $x_k(t)$, $k = 1, \ldots, 4$ on four different figures.

<u>Dynamic Kalman 1-step predictor ($\mathcal{K}$)</u>

(3) After having verified the system observability, insert in the *for* loop the predictor $\mathcal{K}$ defined by

$$\begin{aligned} K(t) &= AP(t)C^T[CP(t)C^T + V_2]^{-1} \\ P(t+1) &= AP(t)A^T + V_1 - K(t)[CP(t)C^T + V_2]K(t)^T \\ \widehat{y}(t|t-1) &= C\widehat{x}(t|t-1) \\ e(t) &= y(t) - \widehat{y}(t|t-1) \\ \widehat{x}(t+1|t) &= A\widehat{x}(t|t-1) + K(t)e(t) \end{aligned}$$

Initialize the variables as: $\widehat{x}(1|0) = 0$, $P(1) = P_1$. Perform the simulation and add to the figures generated at step (2) the plots of the predicted states $\widehat{x}_k(t|t-1)$, $k = 1, \ldots, 4$.

<u>Steady-state Kalman filter ($\mathcal{F}$)</u>

(4) Insert in the *for* loop the filter $\mathcal{F}$ defined by

$$\begin{aligned} K_0(t) &= P(t)C^T[CP(t)C^T + V_2]^{-1} \\ \widehat{x}(t|t) &= \widehat{x}(t|t-1) + K_0(t)\, e(t) \end{aligned}$$

where $P(t)$, $\widehat{x}(t|t-1)$ and $e(t)$ are provided by the Kalman 1-step predictor. Perform the simulation and add to the figures generated at step (2) the plots of the estimated states $\widehat{x}_k(t|t)$, $k = 1, \ldots, 4$.

<u>Steady-state Kalman 1-step predictor ($\mathcal{K}_\infty$)</u>

(5) Insert in the *for* loop the predictor $\mathcal{K}_\infty$ defined by

$$\begin{aligned} \widehat{y}(t|t-1) &= C\widehat{x}(t|t-1) \\ e(t) &= y(t) - \widehat{y}(t|t-1) \\ \widehat{x}(t+1|t) &= A\widehat{x}(t|t-1) + \overline{K}e(t) \end{aligned}$$

where $\overline{K}$ is obtained using the *kalman* Matlab command (outside the *for* loop). Perform the simulation and add to the figures generated at step (2) the plots of the predicted states $\widehat{x}_k(t|t-1)$, $k = 1, \ldots, 4$.

Steady-state Kalman filter ($\mathcal{F}_\infty$)

(6) Insert in the *for* loop the filter $\mathcal{F}_\infty$ defined by

$$\widehat{x}(t|t) = \widehat{x}(t|t-1) + \overline{K}_0 e(t)$$

where $\widehat{x}(t|t-1)$ is the prediction provided by $\mathcal{K}_\infty$ and $\overline{K}_0$ is obtained using the *kalman* Matlab command. Perform the simulation and add to the figures generated at step (2) the plots of the estimated states $\widehat{x}_k(t|t)$, $k = 1, \ldots, 4$.

(Optional) Dynamic Kalman 1-step predictor in predictor/corrector form ($\mathcal{K}_{pc}$)

(7) Insert in the *for* loop the predictor $\mathcal{K}_{pc}$ defined by

$$\begin{gathered}
K_0(t) = P(t)C^T[CP(t)C^T + V_2]^{-1} \\
P_0(t) = [I_n - K_0(t)C]\,P(t) \\
P(t+1) = AP_0(t)A^T + V_1 \\
\widehat{y}(t|t-1) = C\widehat{x}(t|t-1) \\
e(t) = y(t) - \widehat{y}(t|t-1) \\
\widehat{x}(t|t) = \widehat{x}(t|t-1) + K_0(t)e(t) \\
\widehat{x}(t+1|t) = A\widehat{x}(t|t)
\end{gathered}$$

Initialize the variables as: $\widehat{x}(1|0) = 0$, $P(1) = P_1$. Perform the simulation and add to the figures generated at step (2) the plots of the predicted states $\widehat{x}_k(t|t-1)$, $k = 1, \ldots, 4$.

RMSE evaluation

The $RMSE$ can be computed as

$$RMSE = \sqrt{\frac{1}{N - N_0} \sum_{t=N_0+1}^{N} [x_k(t) - \widehat{x}_k(t)]^2}$$

where $\widehat{x}_k(t)$ indicates either $\widehat{x}_k(t|t-1)$ in the case of prediction or $\widehat{x}_k(t|t)$ in the case of filtering, $k = 1, \ldots, 4$, and N_0 is a time after which the filter transient is past.

(8) Evaluate the $RMSE$ errors obtained by the predictors/filters (try with $N_0 = 1$ and $N_0 = 100$).