01RKYQW - Estimation, filtering, and system identification
Report of the sample examination paper

Problem #1. The mean values are removed from the input-output measurements, to obtain zero mean value sequences of length N_{tot}. Then the overall experimental data are partitioned in two datasets: one of length N_e for model identification, the other of length N_v for model validation. For example, the same length may be chosen for each dataset, i.e., $N_e = N_v = N_{tot}/2 = 1000$.

1) The estimation dataset is used to identify ARX, $ARMAX$ and OE models of different orders and delays, and to look for models that guarantee satisfactory characteristics of whiteness of the residuals:
- ARX models of order $n_a = n_b$ and delay $n_k \in [1, 2, 3]$ require $n_a \geq 4$ to have few (no more than 3) values of the autocorrelation function of residuals outside enough the 99% confidence limits;
- $ARMAX$ models of order $n_a = n_b = n_c$ and delay $n_k \in [1, 2, 3]$ require $n_a \geq 2$ to have few (no more than 3) values of the autocorrelation function of residuals outside enough the 99% confidence limits;
- OE models of order $n_f = n_b$ and delay $n_k = 1$ or $n_k \in [2, 3]$ require $n_f \geq 3$ or $n_f \in [3, 4, 5, 7]$, respectively, to have few (no more than 3) values of the autocorrelation function of residuals outside enough the 99% confidence limits.

2) The validation dataset is then used to compare the identified models and to assess their model quality, by minimizing the Root Mean Square Error $RMSE = \sqrt{\frac{1}{N_v - N_0} \sum_{t=N_0+1}^{N_v} [y(t) - \hat{y}(t)]^2}$, where $y(t)$ = measured output, $\hat{y}(t)$ = simulated (or predicted) output and $N_0 = 10$ is chosen as suitable time instant after which the transient is past. Note that, in the case of ARX and $ARMAX$ models, the predicted output provides better performance than the simulated output, since it exploits more information. For this reason, the values of the $RMSE$ using the predicted output only are here reported for all the models:

Identified model	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$ARX(n_a = k, n_b = k, n_k = 1)$	0.0691	0.0664	0.0670	0.0660	0.0638	0.0617	0.0603
$ARX(n_a = k, n_b = k, n_k = 2)$	0.0688	0.0647	0.0668	**0.0656**	0.0638	0.0617	0.0603
$ARX(n_a = k, n_b = k, n_k = 3)$	0.0680	0.0653	0.0662	0.0656	0.0639	0.0618	0.0605
$ARMAX(n_a = k, n_b = k, n_c = k, n_k = 1)$	0.0698	0.0540	0.0500	0.0504	0.0500	0.0521	0.0510
$ARMAX(n_a = k, n_b = k, n_c = k, n_k = 2)$	0.0697	0.0534	0.0536	0.0502	0.0552	0.0508	0.0524
$ARMAX(n_a = k, n_b = k, n_c = k, n_k = 3)$	0.0681	**0.0532**	0.0510	0.0508	0.0510	0.0549	0.0519
$OE(n_b = k, n_f = k, n_k = 1)$	0.1234	0.0584	**0.0504**	0.0507	0.0499	0.0507	0.0511
$OE(n_b = k, n_f = k, n_k = 2)$	0.1167	0.0976	0.0505	0.0506	0.0509	0.0766	0.0507
$OE(n_b = k, n_f = k, n_k = 3)$	0.1115	0.0538	0.0514	0.0511	0.0511	0.0737	0.0516

For ARX models, the best trade-off between $RMSE$ and model complexity $n = n_a + n_b$ that guarantees satisfactory characteristics of whiteness of the residuals is given by the $ARX(4, 4, 2)$, with $RMSE = 0.0656$ and $n = 8$.
For $ARMAX$ models, the best trade-off between $RMSE$ and model complexity $n = n_a + n_b + n_c$ that guarantees satisfactory characteristics of whiteness of the residuals is the $ARMAX(2, 2, 2, 3)$, with $RMSE = 0.0532$ and $n = 6$.
For OE models, the best trade-off between $RMSE$ and model complexity $n = n_b + n_f$ that guarantees satisfactory characteristics of whiteness of the residuals is given by the $OE(3, 3, 1)$, with $RMSE = 0.0504$ and $n = 6$.
In summary, the best trade-off between $RMSE$ and model complexity n that at the same time guarantees satisfactory characteristics of whiteness of the residuals is provided by the $OE(3, 3, 1)$.

3) Using all the experimental data, the following parameters of an $ARX(3, 3, 1)$ model have been estimated by means of the standard Least-Squares algorithm:

$$\hat{\theta}_1 = \hat{a}_1 = -0.6566,\ \hat{\theta}_2 = \hat{a}_2 = -0.3219,\ \hat{\theta}_3 = \hat{a}_3 = 0.1074,\ \hat{\theta}_4 = \hat{b}_1 = 0.0195,\ \hat{\theta}_5 = \hat{b}_2 = 0.0328,\ \hat{\theta}_6 = \hat{b}_3 = 0.0832$$

Assuming that the output measurements are corrupted by an energy-bounded noise whose 2-norm is less than 4, the following Estimate Uncertainty Intervals EUI^2 are derived:

$$\hat{\theta}_1 = \hat{a}_1 \in [-1.9391, 0.6260];\ \hat{\theta}_2 = \hat{a}_2 \in [-1.8080, 1.1641];\ \hat{\theta}_3 = \hat{a}_3 \in [-1.0511, 1.2658];$$
$$\hat{\theta}_4 = \hat{b}_1 \in [-0.7246, 0.7636];\ \hat{\theta}_5 = \hat{b}_2 \in [-0.9347, 1.0004];\ \hat{\theta}_6 = \hat{b}_3 \in [-0.6985, 0.8650]$$

Since the fitting error $||y - \Phi\hat{\theta}|| = 3.05$ is less than the 2-norm noise bound, the Parameter Uncertainty Intervals PUI^2 are given by:

$$\hat{\theta}_1 = \hat{a}_1 \in [-1.4853, 0.1721];\ \hat{\theta}_2 = \hat{a}_2 \in [-1.2821, 0.6382];\ \hat{\theta}_3 = \hat{a}_3 \in [-0.6411, 0.8558];$$
$$\hat{\theta}_4 = \hat{b}_1 \in [-0.4613, 0.5003];\ \hat{\theta}_5 = \hat{b}_2 \in [-0.5924, 0.6580];\ \hat{\theta}_6 = \hat{b}_3 \in [-0.4219, 0.5884]$$

Assuming that the output measurements are corrupted by an amplitude-bounded noise whose ∞-norm is less than 0.1, the following Estimate Uncertainty Intervals EUI^∞ are derived:

$$\hat{\theta}_1 = \hat{a}_1 \in [-1.7464, 0.4332];\ \hat{\theta}_2 = \hat{a}_2 \in [-1.6435, 0.9996];\ \hat{\theta}_3 = \hat{a}_3 \in [-0.8615, 1.0762];$$
$$\hat{\theta}_4 = \hat{b}_1 \in [-0.3781, 0.4171];\ \hat{\theta}_5 = \hat{b}_2 \in [-0.5767, 0.6423];\ \hat{\theta}_6 = \hat{b}_3 \in [-0.3737, 0.5402]$$

Problem #2. First the steady-state Kalman filter $\mathcal{F}_\infty$ and the dynamic Kalman 1-step predictor in predictor-corrector form $\mathcal{K}_{pc}$ are designed, assuming $x(1) = [50, 100, 50]$ as initial state of the system $\mathcal{S}_2$.

2) The state estimates provided by $\mathcal{F}_\infty$ and $\mathcal{K}_{pc}$ have been compared by evaluating the Root Mean Square Errors:

$$RMSE_k = \sqrt{\frac{1}{N' - N'_0} \sum_{t=N'_0+1}^{N'} \left[x_k(t) - \hat{x}_k(t)\right]^2}, \quad k = 1, \ldots, 3$$

where $\hat{x}_k(t)$ is the estimate of the state $x_k(t)$ and $N'_0 = 100$ is chosen as suitable time instant after which the filter or predictor transient is past.

Note that the values of $RMSE$ depend on the realizations of the white noises $v_1(t)$ and $v_2(t)$. For example:

- using the dynamic Kalman 1-step predictor $\mathcal{K}_{pc}$, $RMSE = \begin{bmatrix} 0.0568 \\ 0.1114 \\ 0.0552 \end{bmatrix}$;
- using the steady-state Kalman filter $\mathcal{F}_\infty$, $RMSE = \begin{bmatrix} 0.0561 \\ 0.1113 \\ 0.0556 \end{bmatrix}$.

Note moreover that, if $N'_0 = 0$ is chosen, then:

- using the dynamic Kalman 1-step predictor $\mathcal{K}_{pc}$, $RMSE = \begin{bmatrix} 3.4358 \\ 7.2297 \\ 3.7862 \end{bmatrix}$;
- using the steady-state Kalman filter $\mathcal{F}_\infty$, $RMSE = \begin{bmatrix} 3.6312 \\ 7.5607 \\ 3.9285 \end{bmatrix}$.