System identification, Estimation and Filtering

Consider the following LTI system:

\[
x(t + 1) = Ax(t) + v_1(t) \\
y(t) = Cx(t) + v_2(t)
\]

where

\[
A = \begin{bmatrix}
0.96 & 0.5 & 0.27 & 0.28 \\
-0.125 & 0.96 & -0.08 & -0.07 \\
0 & 0 & 0.85 & 0.97 \\
0 & 0 & 0 & 0.99
\end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & 0 & 0 \end{bmatrix}
\]

\(v_1(t)\) is a white noise with zero mean value and variance \(V_1 = B_v B_v^T\), \(B_v = \sqrt{15}[0.5 \ 0 \ 0 \ 1]^T\), and \(v_2(t)\) is a white noise with zero mean value and variance \(V_2 = 2000\). The noises \(v_1(t)\) and \(v_2(t)\) are uncorrelated. Assume that the initial state \(x(1)\) is a random vector with zero mean value and variance \(P_1 = E[x(1)x(1)^T] = 0.5I_4\).

**Problem:** Design the following predictors/filters:
- Dynamic Kalman 1-step predictor.
- Dynamic Kalman filter.
- Steady-state Kalman 1-step predictor.
- Steady-state Kalman filter.
- (Optional) Dynamic Kalman 1-step predictor in predictor/corrector form.

Compare the estimates provided by these predictors/filters by means of graphical representations and evaluation of the Root Mean Square Error (RMSE).

**Main steps:**

1. Create a Matlab program for the simulation of the system (1). Use a `for` loop in order to implement the simulation. In the program, the noise \(v_1(t)\) can be generated as \(v1(:,t)=Bv.*randn(4,1)\); the noise \(v_2(t)\) can be generated as \(v2(t)=sqrt(V2)*randn\);

2. Simulate the system for \(t = 1, 2, ..., N\), \(N = 2000\) starting from random initial conditions: \(x(1) = \sqrt{0.5} randn(4,1)\). Plot the four state signals \((x_k(t), k = 1, \ldots, 4)\) on four different figures.

Dynamic Kalman 1-step predictor (\(K\))

3. After having verified the system observability, insert in the `for` loop the predictor \(K\) defined by

\[
K(t) = AP(t)C^T[CP(t)C^T + V_2]^{-1} \\
P(t + 1) = AP(t)A^T + V_1 - K(t)[CP(t)C^T + V_2]K(t)^T \\
\hat{y}(t|t - 1) = C\hat{x}(t|t - 1) \\
e(t) = y(t) - \hat{y}(t|t - 1) \\
\hat{x}(t + 1|t) = A\hat{x}(t|t - 1) + K(t)e(t)
\]

Initialize the variables as: \(\hat{x}(1|0) = 0, P(1) = P_1\). Perform the simulation and add to the figures generated at step (2) the plots of the predicted states \(\hat{x}_k(t|t - 1), k = 1, \ldots, 4\).

Steady-state Kalman filter (\(F\))

4. Insert in the `for` loop the filter \(F\) defined by

\[
K_0(t) = P(t)C^T[CP(t)C^T + V_2]^{-1} \\
\hat{x}(t) = \hat{x}(t|t - 1) + K_0(t)e(t)
\]

where \(P(t), \hat{x}(t|t - 1)\) and \(e(t)\) are provided by the Kalman 1-step predictor. Perform the simulation and add to the figures generated at step (2) the plots of the estimated states \(\hat{x}_k(t|t), k = 1, \ldots, 4\).

Steady-state Kalman 1-step predictor (\(K_{\infty}\))

5. Insert in the `for` loop the predictor \(K_{\infty}\) defined by

\[
\hat{y}(t|t - 1) = C\hat{x}(t|t - 1) \\
e(t) = y(t) - \hat{y}(t|t - 1) \\
\hat{x}(t + 1|t) = A\hat{x}(t|t - 1) + K_{\infty}e(t)
\]
where $\mathbf{K}$ is obtained using the \textit{kalman} Matlab command (outside the \textit{for} loop). Perform the simulation and add to the figures generated at step (2) the plots of the predicted states $\hat{x}_k(t|t-1)$, $k = 1, \ldots, 4$.

**Steady-state Kalman filter ($\mathcal{F}_\infty$)**

(6) Insert in the \textit{for} loop the filter $\mathcal{F}_\infty$ defined by

$$\hat{x}(t|t) = \hat{x}(t|t-1) + K_0 e(t)$$

where $\hat{x}(t|t-1)$ is the prediction provided by $\mathcal{K}_\infty$ and $K_0$ is obtained using the \textit{kalman} Matlab command. Perform the simulation and add to the figures generated at step (2) the plots of the estimated states $\hat{x}_k(t|t)$, $k = 1, \ldots, 4$.

(Optional) Dynamic Kalman 1-step predictor in predictor/corrector form ($\mathcal{K}_{pc}$)

(7) Insert in the \textit{for} loop the predictor $\mathcal{K}_{pc}$ defined by

$$K_0(t) = P(t) C^T [CP(t)C^T + V_2]^{-1}$$

$$P_0(t) = [I_n - K_0(t)C] P(t)$$

$$P(t+1) = AP(t)A^T + V_1$$

$$\hat{y}(t|t-1) = C\hat{x}(t|t-1)$$

$$e(t) = y(t) - \hat{y}(t|t-1)$$

$$\hat{x}(t|t) = \hat{x}(t|t-1) + K_0(t)e(t)$$

$$\hat{x}(t+1|t) = A\hat{x}(t|t)$$

Initialize the variables as: $\hat{x}(1|0) = 0$, $P(1) = P_1$. Perform the simulation and add to the figures generated at step (2) the plots of the predicted states $\hat{x}_k(t|t-1)$, $k = 1, \ldots, 4$.

**RMSE evaluation**

The \textit{RMSE} can be computed as

$$\text{RMSE} = \sqrt{\frac{1}{N - N_0} \sum_{t=N_0+1}^{N} [x_k(t) - \hat{x}_k(t)]^2}$$

where $\hat{x}_k(t)$ indicates either $\hat{x}_k(t|t-1)$ in the case of prediction or $\hat{x}_k(t|t)$ in the case of filtering, $k = 1, \ldots, 4$, and $N_0$ is a time after which the filter transient is past.

(8) Evaluate the \textit{RMSE} errors obtained by the predictors/filters (try with $N_0 = 1$ and $N_0 = 100$).