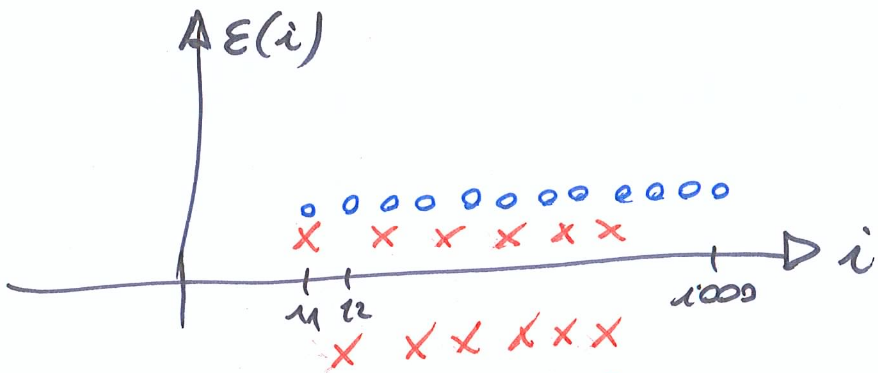
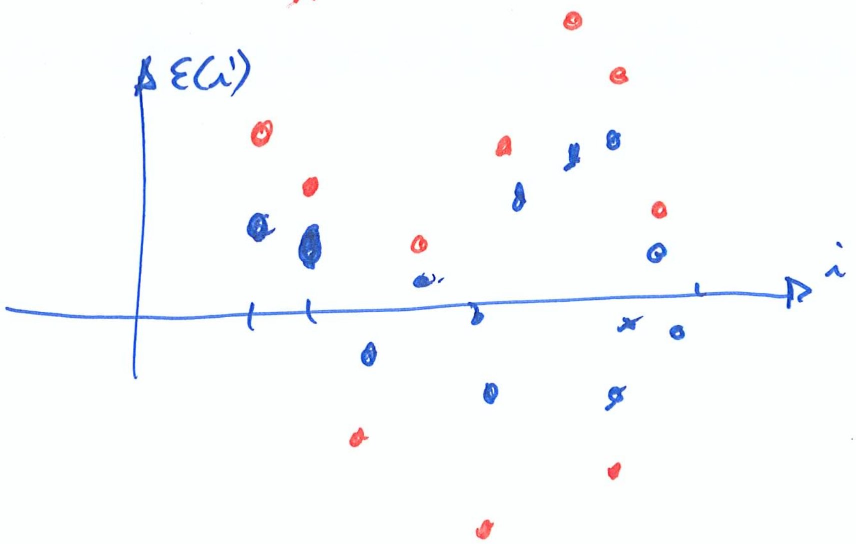
$\hat{y}(11|10) \longrightarrow \varepsilon(11) = y(11) - \hat{y}(11|10)$

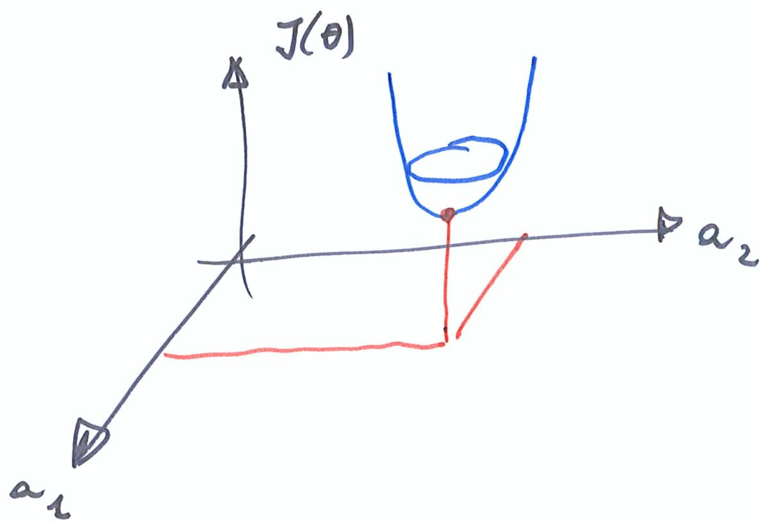
$\hat{y}(12|11) \longrightarrow \varepsilon(12) = y(12) - \hat{y}(12|11)$

$\vdots$

$\hat{y}(1000|999) \longrightarrow \varepsilon(1000) = y(1000) - \hat{y}(1000|999)$

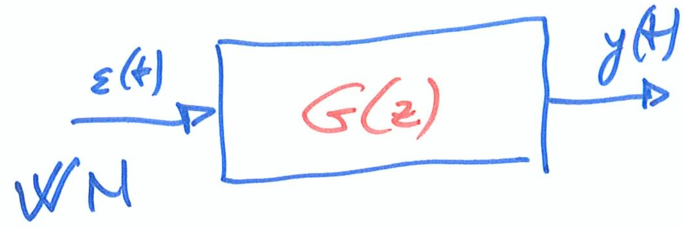




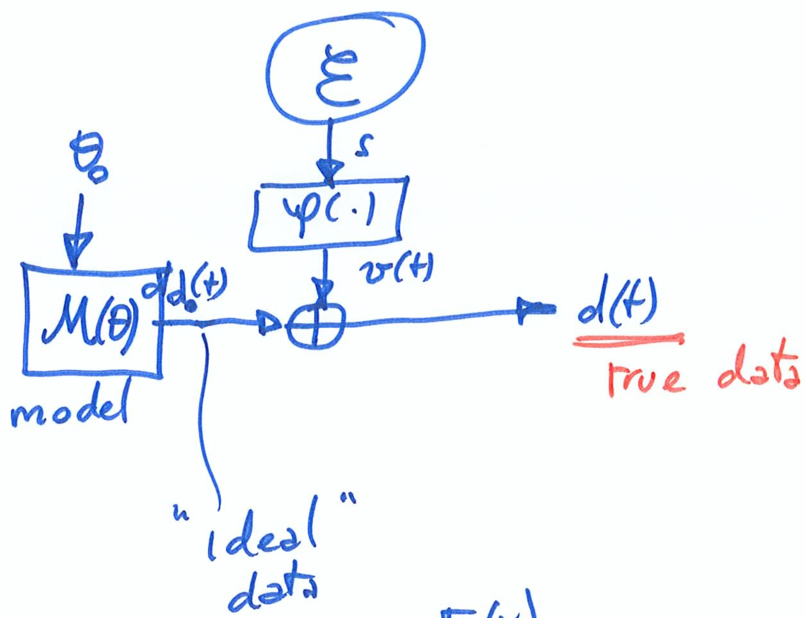


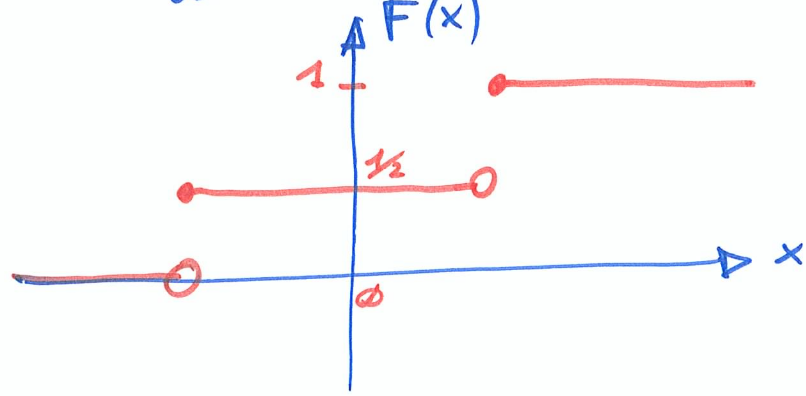
$\theta = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

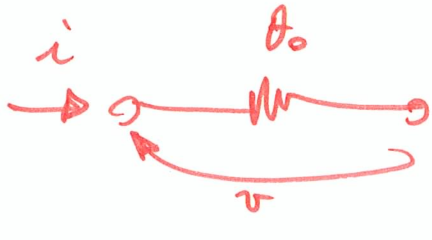
$$y(t) = a_1 y(t-1) + a_2 y(t-2) + a_3 y(t-3) + \ldots + a_n y(t-n) + \varepsilon(t)$$

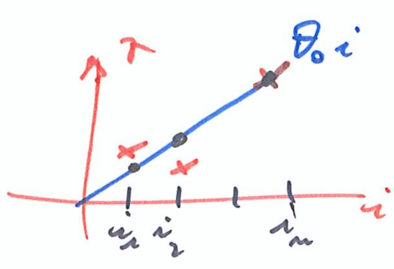


stochastic system = dynamical system driven by a stochastic signal









$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$E[v] = \bar{v} = \begin{bmatrix} E[v_1] \\ E[v_2] \end{bmatrix} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix}$$

$$\Sigma_v = E\left[(v - \bar{v}) \cdot (v - \bar{v})^T \right]$$

Σ_v is symmetric:

$$\Sigma_v = E\left[\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} \right)^T \right] =$$

$$= E\left[\begin{bmatrix} v_1 - \bar{v}_1 \\ v_2 - \bar{v}_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 - \bar{v}_1 \\ v_2 - \bar{v}_2 \end{bmatrix}^T \right] =$$

$$= E\left[\begin{bmatrix} v_1 - \bar{v}_1 \\ v_2 - \bar{v}_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 - \bar{v}_1 , & v_2 - \bar{v}_2 \end{bmatrix} \right] =$$

$$= E\begin{bmatrix} (v_1 - \bar{v}_1)^2 & (v_1 - \bar{v}_1)(v_2 - \bar{v}_2) \\ (v_2 - \bar{v}_2)(v_1 - \bar{v}_1) & (v_2 - \bar{v}_2)^2 \end{bmatrix} =$$

$$= \begin{bmatrix} E[(v_1 - \bar{v}_1)^2] & E[(v_1 - \bar{v}_1)(v_2 - \bar{v}_2)] \\ E[(v_1 - \bar{v}_1)(v_2 - \bar{v}_2)] & E[(v_2 - \bar{v}_2)^2] \end{bmatrix} = \Sigma_v^T$$

$$= \begin{bmatrix} \Sigma_{v_1} \ \sigma_1^2 \geq 0 & \sigma_{v_1 v_2} \ \sigma_{12} \\ \sigma_{v_1 v_2} \ \sigma_{12} & \Sigma_{v_2} \ \sigma_2^2 \geq 0 \end{bmatrix}$$

Σ_{v} is <u>positive semidefinite</u> $(\Sigma_{v} \geq 0)$,

$$x^T \Sigma_{v} x \geq 0, \quad \forall x \in \mathbb{R}^m$$

$$x^T E\left[(v-\bar{v})(v-\bar{v})^T\right] x = E\Big[\underbrace{x^T (v-\bar{v})}_{a} \underbrace{(v-\bar{v})^T x}_{b}\Big]$$

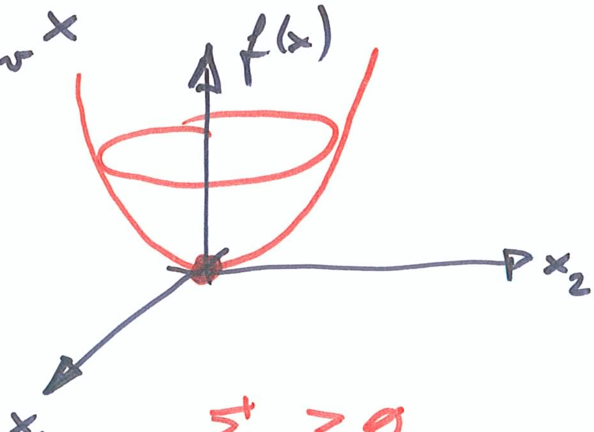
$$a = x^T (v-\bar{v})$$

$$a^T = \left[x^T (v-\bar{v})\right]^T = (v-\bar{v})^T \cdot (x^T)^T = (v-\bar{v})^T \cdot x = b \equiv a$$

$$x^T \Sigma_{v} x = E\Big[\underbrace{\left((v-\bar{v})^T \cdot x\right)^2}_{\geq 0}\Big] \geq 0$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \Sigma_{v} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

$$f(x) = x^T \Sigma_{v} x$$



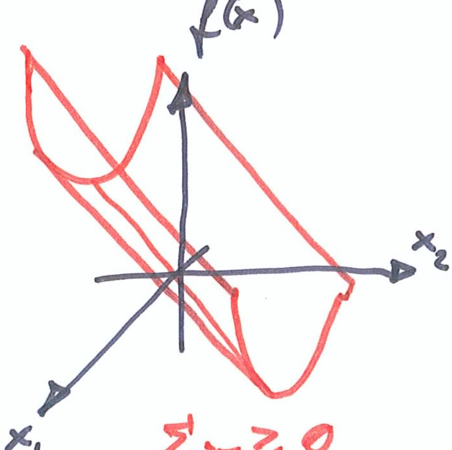
$\Sigma_{v} > 0$

Σ_{v} positive definite

$\Downarrow$

$\det(\Sigma_{v}) > 0$

$\lambda_i > 0, \ \forall i$

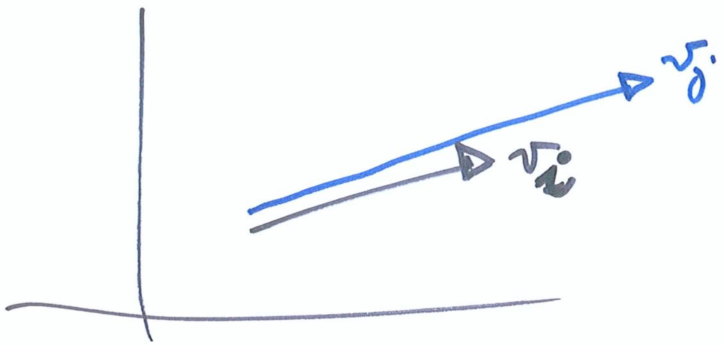


$\Sigma_{v} \geq 0$

Σ_{v} positive <u>semidefinite</u>

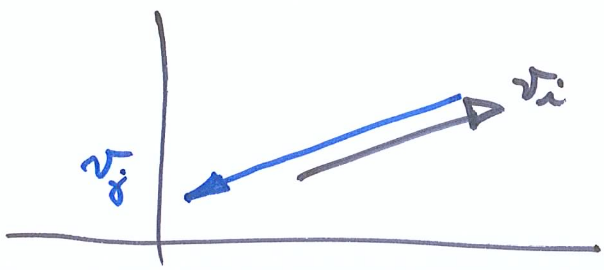
$\det(\Sigma_{v}) \geq 0$

$\lambda_i \geq 0, \ \forall i$



$\alpha > 0$

$$v_j = \alpha v_i + \underbrace{\cancel{\beta}}_{=0}$$



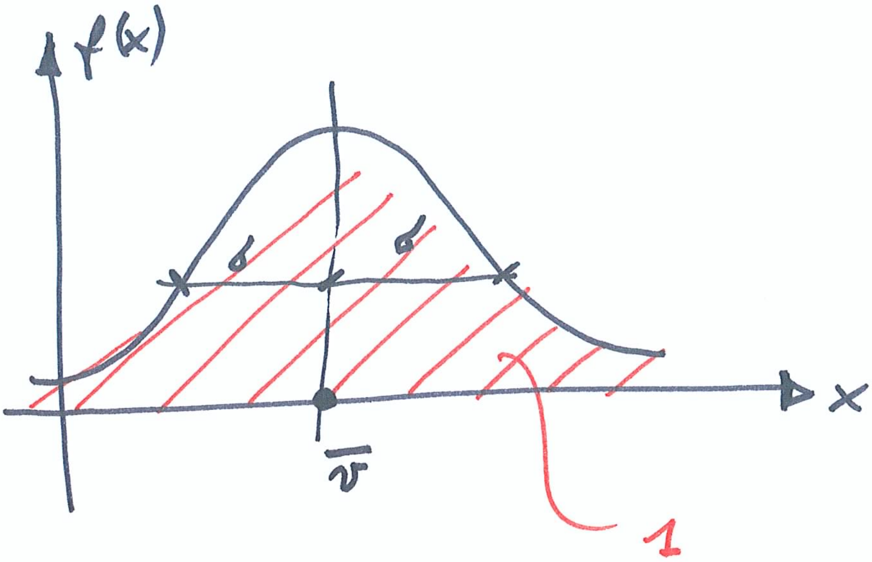
$\alpha < 0$

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \frac{\alpha \sigma_i^2}{\sqrt{\sigma_i^2} \cdot \sqrt{\alpha^2 \sigma_i^2}} = \frac{\alpha \cancel{\sigma_i^2}}{\cancel{\sqrt{\sigma_i^2}} \sqrt{\alpha^2} \cancel{\sqrt{\sigma_i^2}}}$$

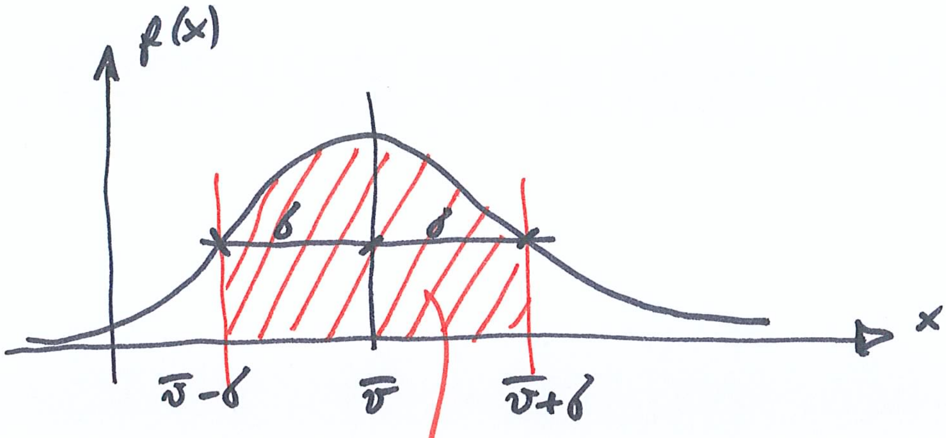
$$= \frac{\alpha}{\sqrt{\alpha^2}} = \frac{\alpha}{|\alpha|} = \begin{cases} +1, & \alpha > 0 \\ -1, & \alpha < 0 \end{cases}$$

v contains i.i.d. components

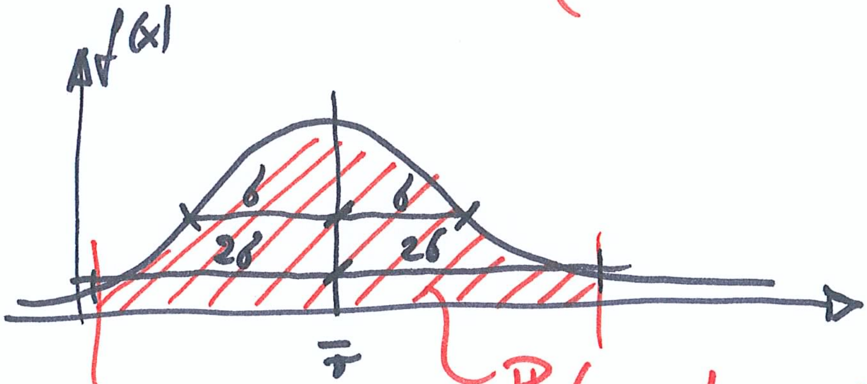
i: independent
i: identically
d: distributed



$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$



$$\mathbb{P}(\bar{v} - \sigma \le v \le \bar{v} + \sigma) = 0{,}683$$



$$\mathbb{P}(\bar{v} - 2\sigma \le v \le \bar{v} + 2\sigma) = 0{,}954$$

$$\underbrace{\mathrm{Var}[\hat{\theta}^{(1)}]}_{\square} \le \underbrace{\mathrm{Var}[\hat{\theta}^{(2)}]}_{\square}, \qquad \forall \hat{\theta}^{(2)} \ne \hat{\theta}^{(1)}$$

$$A = \mathrm{Var}[\hat{\theta}^{(2)}] - \mathrm{Var}[\hat{\theta}^{(1)}] \ge 0$$

$$x^T A x \ge 0, \ \forall x$$

$$y(t) = \theta_1 u_1(t) + \theta_2 u_2(t) + \dots + \theta_m u_m(t), \qquad t = 1, 2, \dots, N$$

$$\begin{cases} y(1) = u_1(1)\,\theta_1 + u_2(1)\,\theta_2 + \dots + u_m(1)\,\theta_m \\ y(2) = u_1(2)\,\theta_1 + u_2(2)\,\theta_2 + \dots + u_m(2)\,\theta_m \\ \vdots \\ y(N) = u_1(N)\,\theta_1 + u_2(N)\,\theta_2 + \dots + u_m(N)\,\theta_m \end{cases} \qquad m \ll N$$

$$\Updownarrow$$

$$Y = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}, \qquad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{bmatrix} \in \mathbb{R}^m \qquad \varphi(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m$$

regression vector

unknown

$$\begin{cases} y(1) = \varphi(1)^T \cdot \theta \\ y(2) = \varphi(2)^T \cdot \theta \\ \vdots \\ y(N) = \varphi(N)^T \cdot \theta \end{cases} \iff \boxed{Y = \underbrace{\begin{bmatrix} \varphi(1)^T \\ \varphi(2)^T \\ \vdots \\ \varphi(N)^T \end{bmatrix}}_{\Phi} \cdot \theta = \Phi \cdot \theta}$$

Known

Φ = regression matrix

Similar problem:

$A x \cong b$ overdetermined problem

$N \geq m$

$$\theta^* = \arg\min_{\theta \in \mathbb{R}^n} J(\theta), \quad J(\theta) = \sum_{t=1}^{N} \varepsilon(t)^2$$

$$\varepsilon(t) = y(t) - \varphi(t)^T \theta$$

$$\varphi(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}$$

$$\frac{dJ(\theta)}{d\theta} = \text{gradient of } J(\theta) = \left[\frac{dJ(\theta)}{d\theta_1} \cdots \frac{dJ(\theta)}{d\theta_n} \right]$$

$$\frac{dJ(\theta)}{d\theta_i} = \cancel{\frac{d\sum_{t=1}^{N}\varepsilon(t)}{d\theta_i}} \; \frac{d\sum_{t=1}^{N}\varepsilon(t)^2}{d\theta_i} = \sum_{t=1}^{N} \frac{d\varepsilon(t)^2}{d\theta_i} =$$

$$= \sum_{t=1}^{N} \frac{d\left[y(t) - \varphi(t)^T\theta\right]^2}{d\theta_i} =$$

$$= \sum_{t=1}^{N} 2\left[y(t) - \varphi(t)^T\theta\right](-1)\,u_i(t) = 0, \quad i = 1, 2, \dots, n$$

$$\sum_{t=1}^{N} \left[y(t) - \varphi(t)^T\theta\right] u_i(t) = 0, \quad i = 1, 2, \dots, n$$

$$\begin{cases} \sum_{t=1}^{N} \left[y(t) - \varphi(t)^T\theta\right] u_1(t) = 0 \\ \quad " \quad [\quad\quad] \, u_2(t) = 0 \\ \quad \vdots \\ \quad " \quad [\quad\quad] \, u_n(t) = 0 \end{cases} \iff (-2)\sum_{t=1}^{N} \underbrace{\left[y(t) - \varphi(t)^T\theta\right]\varphi(t)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{0}$$

$$\sum_{t=1}^{N} y(t)\varphi(t) = \sum_{t=1}^{N} \varphi(t)^T\theta\,\varphi(t) = \sum_{t=1}^{N} \overbrace{\varphi(t)\,\varphi(t)^T\theta}$$

$$\cancel{\sum_{t=1}^{N} y(t)\,\varphi(t)^T = \sum_{t=1}^{N} \varphi(t)^T\theta\,\varphi(t)^T = \sum_{t=1}^{N} \varphi(t)^T\varphi(t)^T\theta}$$

$$\underbrace{\sum_{t=1}^{N}\left[\varphi(t)\varphi(t)^T\right]}\cdot\theta = \underbrace{\sum_{t=1}^{N}\varphi(t)\cdot y(t)}$$

$$\underbrace{\sum_{t=1}^{N} \left[\varphi(t)\,\varphi(t)^T\right]}\cdot\theta = \underbrace{\sum_{t=1}^{N} \varphi(t)\cdot y(t)}$$

Normal Equations

$$A \qquad \theta = \qquad b$$

If $\det(A) \neq 0 \Rightarrow \exists$ a unique solution $\theta = A^{-1} b$

$$\text{Hessian matrix} = \frac{d^2 J(\theta)}{d\theta^2} = \frac{d}{d\theta}\left[\frac{dJ(\theta)}{d\theta}\right]^T$$

$$= \frac{d}{d\theta}\left[\sum_{t=1}^{N}(-2)\left[y(t)\varphi(t) - \varphi(t)^T\theta\,\varphi(t)\right]\right] =$$

$$\left(\left[\frac{dJ(\theta)}{d\theta}\right]^T = \sum_{t=1}^{N}(-2)\left[y(t) - \varphi(t)^T\theta\right]\underbrace{\varphi(t)}\right)$$

$$= \frac{d}{d\theta}\left[-\sum_{t=1}^{N}(-2)\,\varphi(t)\,\varphi(t)^T\theta\right] = +2\sum_{t=1}^{N}\varphi(t)\,\varphi(t)^T$$

$$\frac{dJ(\theta)}{d\theta_i} = \sum_{t=1}^{N} 2\left[y(t) - \varphi(t)^T\theta\right](-1)\, u_i(t) = 0 \quad , \quad i = 1, \dots, n$$

$$\frac{dJ(\theta)}{d\theta} = \text{gradient of } J(\theta) = \left[\frac{dJ(\theta)}{d\theta_1}, \dots, \frac{dJ(\theta)}{d\theta_n}\right] =$$

$$= (-2)\sum_{t=1}^{N}\left[y(t) - \varphi(t)^T\theta\right]\underbrace{\left[u_1(t), \dots, u_n(t)\right]}_{\varphi(t)^T} =$$

$$= (-2)\sum_{t=1}^{N}\left[y(t) - \varphi(t)^T\theta\right]\varphi(t)^T$$

$$\left[\frac{dJ(\theta)}{d\theta}\right]^T = \left[(-2)\sum_{t=1}^{N}\left[y(t) - \varphi(t)^T\theta\right]\varphi(t)^T\right]^T =$$

$$= (-2)\sum_{t=1}^{N}\left[y(t) - \varphi(t)^T\theta\right]\varphi(t) =$$

$$= (-2)\sum_{t=1}^{N} y(t)\varphi(t) + 2\sum_{t=1}^{N}\varphi(t)^T\theta\,\varphi(t) =$$

$$= (-2)\sum_{t=1}^{N} y(t)\varphi(t) + 2\sum_{t=1}^{N}\varphi(t)\varphi(t)^T\theta$$

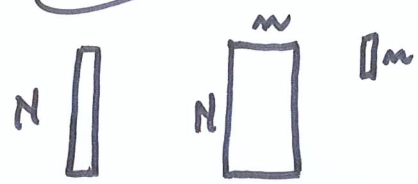
In a matrix form ($N \gg m$)

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix} \in \mathbb{R}^m, \qquad \varphi(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m = \text{regression vector}$$

$$Y = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix} \in \mathbb{R}^N, \qquad \Phi = \begin{bmatrix} \varphi(1)^T \\ \vdots \\ \varphi(N)^T \end{bmatrix} = \text{regression matrix} \in \mathbb{R}^{N \times m}$$

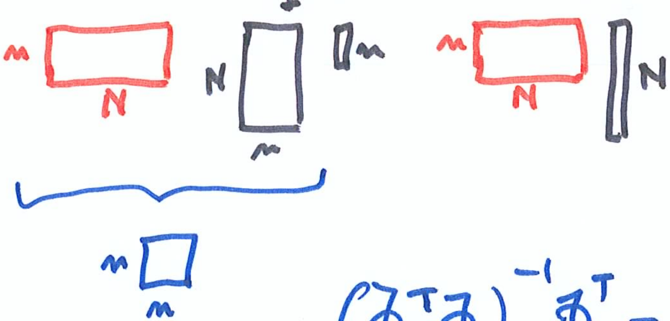
$$Y = \Phi\,\theta \quad \Rightarrow \quad \theta = ?$$

known (Y, Φ) unknown (θ)



By premultiplying for Φ^T

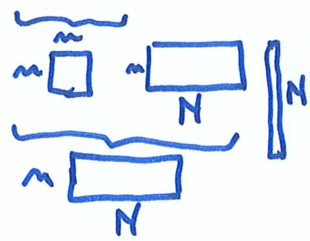
$$\Phi^T \Phi\, \theta = \Phi^T Y$$



$\xrightarrow{}$ if $\Phi^T\Phi$ is invertible ($\det(\Phi^T\Phi) \neq 0$)

by premultiplying for $(\Phi^T\Phi)^{-1}$:

$$\boxed{\theta = (\Phi^T\Phi)^{-1}\Phi^T Y}$$



$(\Phi^T\Phi)^{-1}\Phi^T$ = pseudoinverse of Φ

If Φ is invertible (if $m = N$, and $\det \Phi \neq 0$), then there exists Φ^{-1} $\Rightarrow$ $(\Phi^T\Phi)^{-1}\Phi^T = (\Phi)^{-1}\underbrace{(\Phi^T)^{-1}\Phi^T}_{I_m} = \Phi^{-1}$

Under MATLAB

pinv(Phi) $\longrightarrow$ $[(\text{Phi})^T\,\text{Phi}]^{-1}\,\text{Phi}^T$

$\hat{\theta}_{LS}$ = pinv(Phi) * Y

= Phi \ Y

Example: if v is a vector of uncorrelated noises $\Rightarrow$

$$\Sigma_v = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_N^2 \end{bmatrix}$$

$$Q^* = \Sigma_v^{-1} = \begin{bmatrix} 1/\sigma_1^2 & & & 0 \\ & 1/\sigma_2^2 & & \\ & & \ddots & \\ 0 & & & 1/\sigma_N^2 \end{bmatrix} = \frac{1}{\det \Sigma_v} \mathrm{Adj}[\Sigma_v]$$

$$J_{WLS}(\theta^*) = \varepsilon^T Q^* \varepsilon = \sum_{t=1}^{N} \frac{\varepsilon(t)^2}{\sigma_t^2}$$

Vector norms

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$$

$\|x\|_2$ = Euclidean norm of $x = \sqrt{\sum_{i=1}^{m} x_i^2}$

$\|x\|_\infty$ = infinity norm of $x = \max_{i=1,\dots,m} |x_i|$

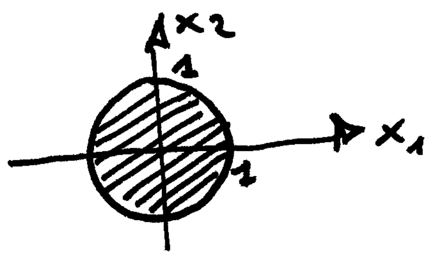
$\mathcal{B}^2$ = unitary ball in the 2-norm =

$= \{x \in \mathbb{R}^m : \|x\|_2 \le 1\} =$

$= \{x \in \mathbb{R}^m : \sqrt{x_1^2 + x_2^2} \le 1\} =$

$= \{x \in \mathbb{R}^m : x_1^2 + x_2^2 \le 1\} \subset \mathcal{B}^\infty$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



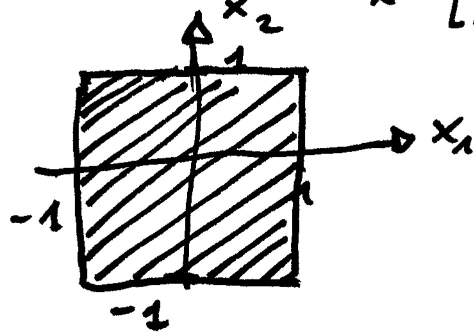
$\mathcal{B}^\infty$ = unitary ball in the ∞-norm =

$= \{x \in \mathbb{R}^m : \|x\|_\infty \le 1\} =$

$= \{x \in \mathbb{R}^m : \max_{i=1,\dots,m} |x_i| \le 1\} =$

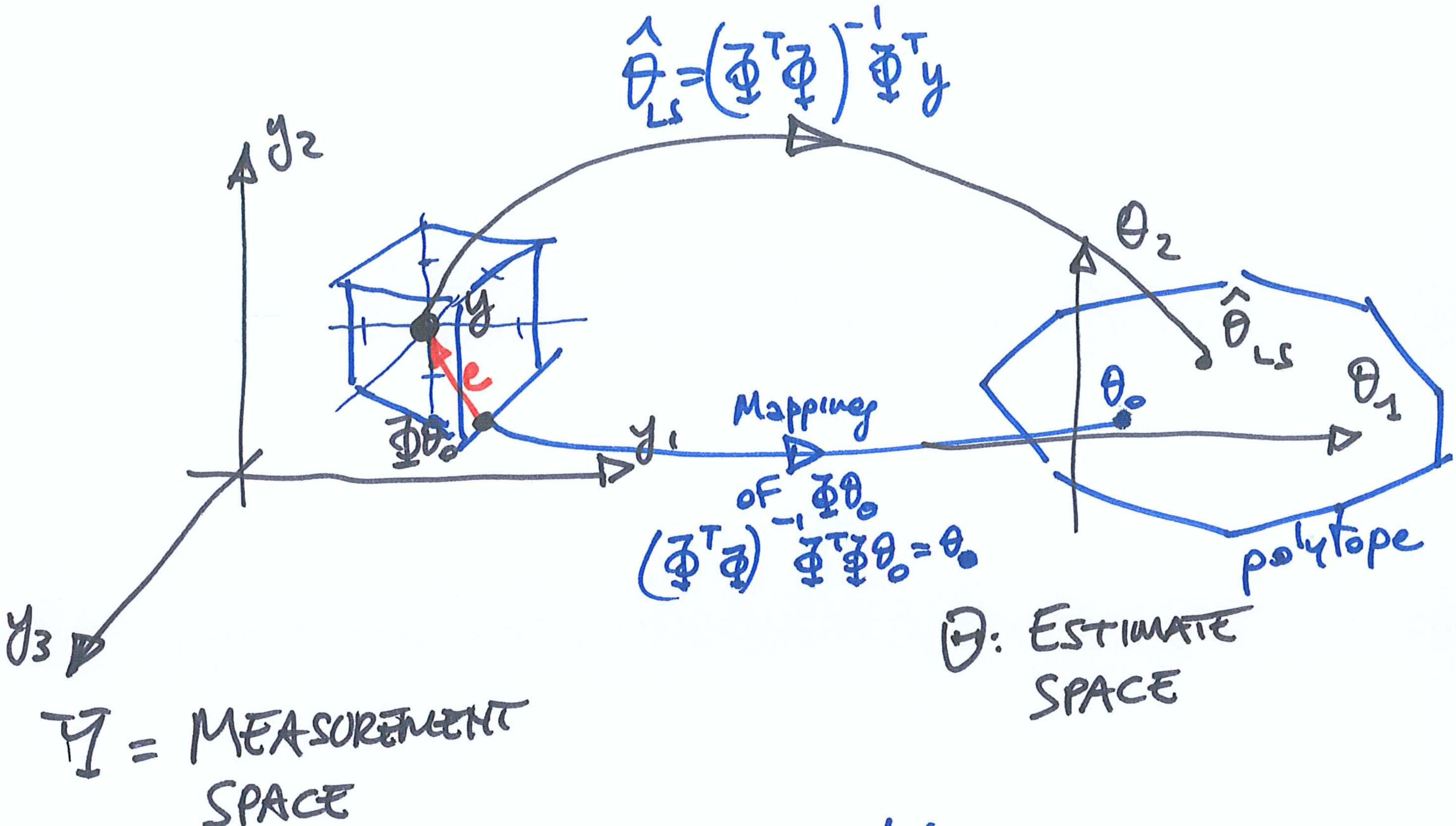
$= \left\{x \in \mathbb{R}^m : \begin{cases} |x_1| \le 1 \\ |x_2| \le 1 \\ \vdots \\ |x_m| \le 1 \end{cases}\right\} \supset \mathcal{B}^2$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



Set Membership Framework

- Linear estimation problem: $y = \Phi \theta_0 + e$
- $e \in B$ = suitable set of noise (For example, B^∞)
- for example, $N=3$, $n=2$



MUS = Measurement Uncertainty Set

$= \{\tilde{y} : \tilde{y} - y \in B\}$

Linear Estimation problem in the set Membership framework

- "A priori" information
  - we know $\underline{\underline{\Phi}}$ such that: $y = \Phi \theta_0 + e$
  - the noise $e \in \underline{\underline{\mathcal{B}_e}}$ known
- "A posteriori" information
  - we know a measurement vector $\underline{\underline{y}}$
- Goal: Find a suitable estimate

$$\hat{\theta} = \underset{\text{estimator}}{\psi}(y) \cong \underset{\text{unknown}}{\theta_0}$$

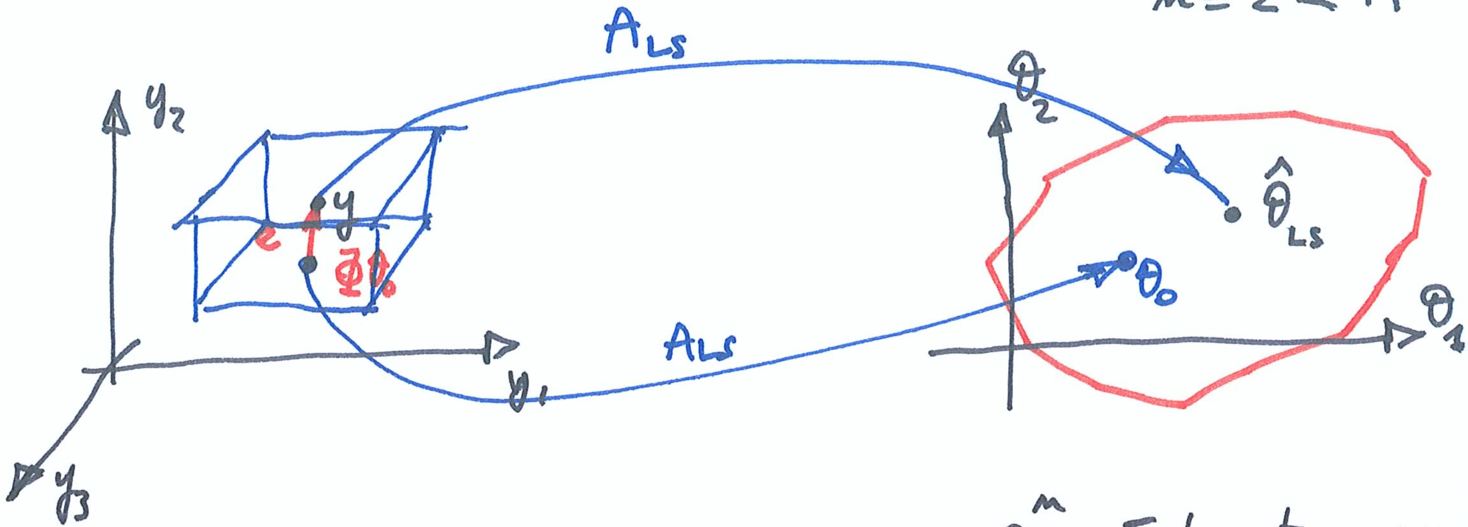
and the uncertainty intervals about $\hat{\theta}$

Possible solution: apply the least square estimator

$$\hat{\theta}_{LS} = A_{LS}\, y = \underbrace{(\Phi^T \Phi)^{-1} \Phi^T}_{\substack{\text{pseudoinverse} \\ \text{of } \Phi}} y$$

$N = 3$

$m = 2 < N$



$\mathbb{R}^N$ = Measurement space

$\mathbb{R}^m$ = Estimate space

IF $e \in B_e^\infty \Rightarrow \Phi\theta_0 \in$ Cube B_e^∞ centered around y

$$y \oplus B_e^\infty = MUS^\infty$$

direct sum

= Measurent Uncertainty set

$$= \{\tilde{y} : \|\tilde{y} - y\|_\infty \le \varepsilon\}$$

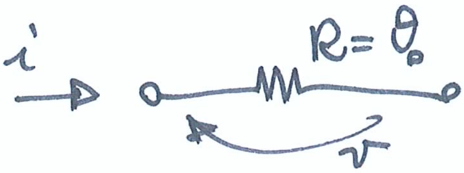
Using $A_{LS} = (\Phi^T\Phi)^{-1}\Phi^T \Rightarrow$ if we apply A_{LS} to $\Phi\theta_0$, then

$$A_{LS} \cdot \Phi\theta_0 = \underbrace{(\Phi^T\Phi)^{-1}\Phi^T \cdot \Phi}_{I}\,\theta_0 = \theta_0$$

IF we apply A_{LS} to the MUS^∞, we obtain a set called

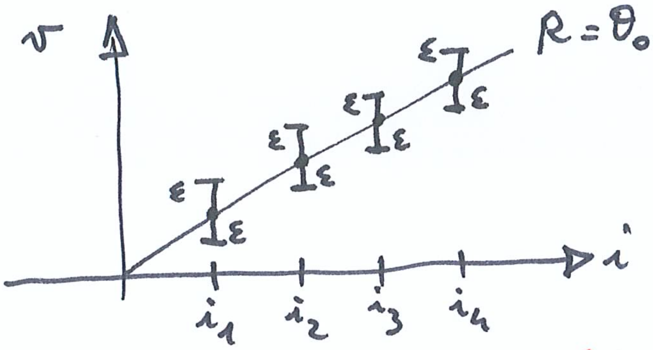
Estimate Uncertainty set $= EUS^\infty \ni \theta_0$

Example :

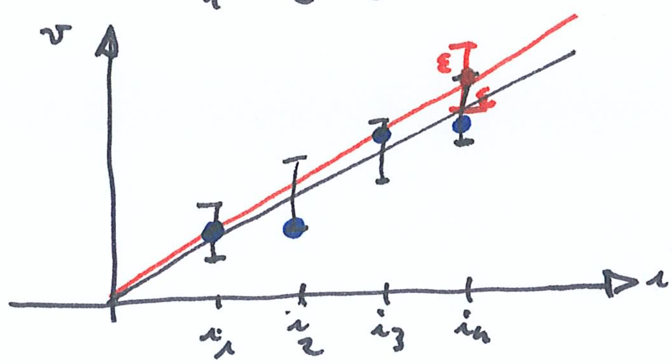


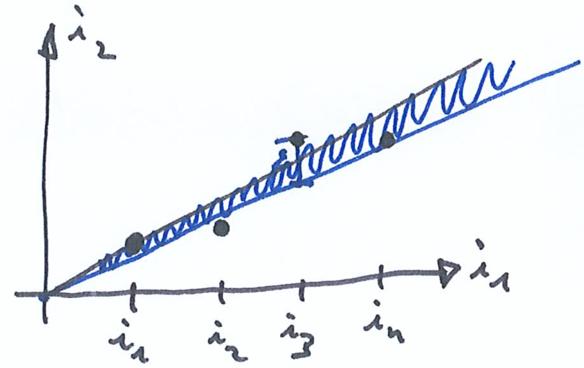
$$\begin{cases} v_1 = R i_1 + e_1 \\ v_2 = R i_2 + e_2 \\ \vdots \\ v_N = R i_N + e_N \end{cases}$$

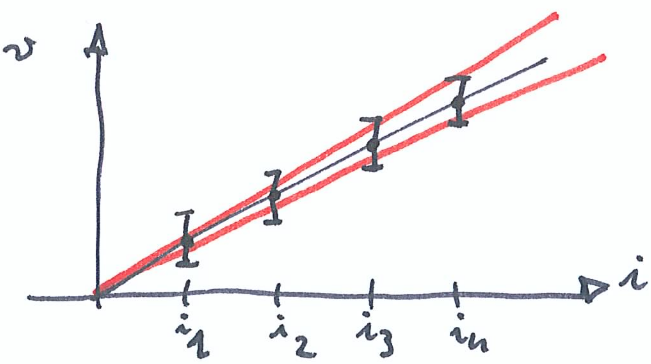
$$y = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}, \quad \Phi = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} \Rightarrow \begin{aligned} y &= \Phi \cdot R + e \\ &= \Phi \cdot \theta_0 + e \end{aligned}$$



$$\|e\|_\infty \leq \varepsilon$$

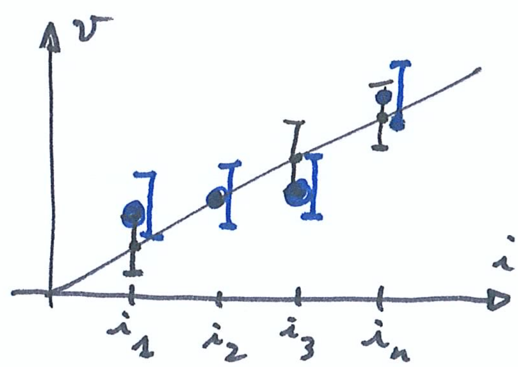






worst-case for

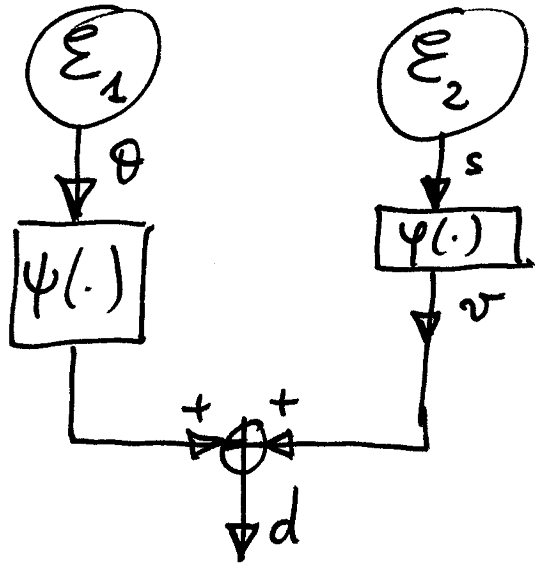
noise-free case



best-case $\Rightarrow$

$\hat{R} \equiv \theta_0$

boundary-visiting noise

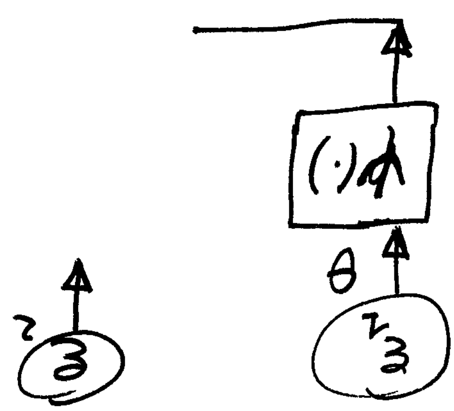


$$\mathbb{P}(A,B) = \mathbb{P}(A|B)\cdot\mathbb{P}(B) = \mathbb{P}(B|A)\cdot\mathbb{P}(A) \Rightarrow \mathbb{P}(A|B) = \frac{\mathbb{P}(A,B)}{\mathbb{P}(B)}$$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A,B)}{\mathbb{P}(A)}$$

BAYES's RULE

$$v \sim \mathcal{N}(\bar{v}, \sigma_v^2) = c' \exp\left(-\frac{1}{2\sigma_v^2}(v-\bar{v})^2\right)$$



$$\hat{v}_2 = \frac{E[v_1 v_2]}{Var[v_1]} v_1$$

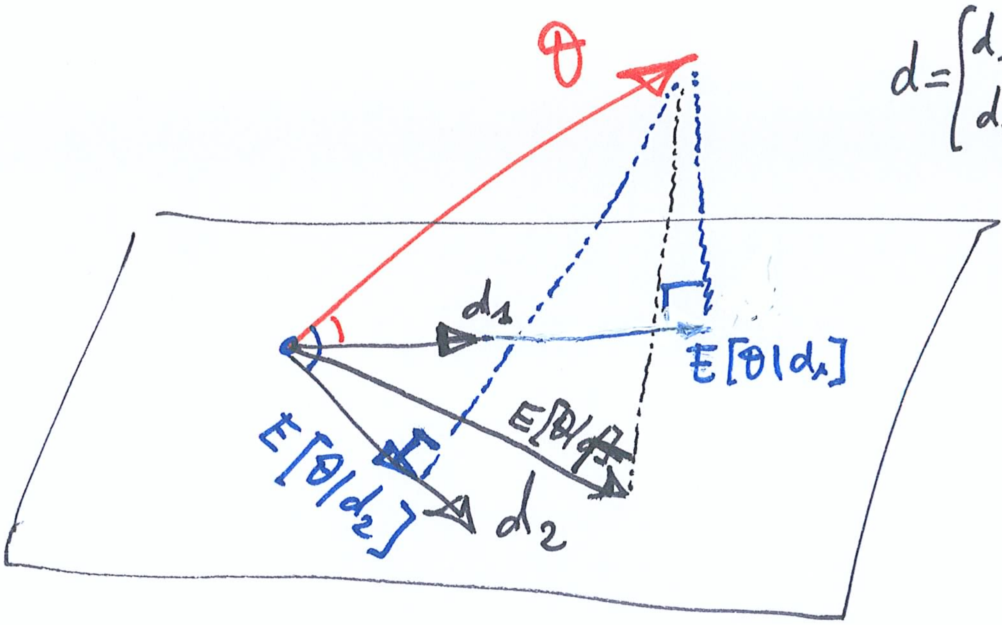
$$\|\hat{v}_2\| = \left\| \frac{E[v_1 v_2]}{Var[v_1]} v_1 \right\| =$$

$$= \frac{|E[v_1 v_2]|}{\|v_1\|^{\cancel{2}}} \cancel{\|v_1\|} = \frac{|E[v_1 v_2]|}{\|v_1\|}$$

$$\|\hat{v}_2\|^2 = \frac{(E[v_1 v_2])^2}{\|v_1\|^2}$$

θ is scalar r.v.

$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ is a 2-dimensional random variable



Deterministic framework for state estimation

Let us consider a dynamic LTI system in the discrete-time domain:

$$
\begin{aligned}
x(t+1) &= A\,x(t) + B\,u(t) && \text{state equation} \\
y(t) &= C\,x(t) + \cancel{D\,u(t)} && \text{output equation}
\end{aligned}
$$

physically-realizable system: $D = 0$

The matrices A, B, C are known

The input $u(t)$ is known $\forall t$

The output $y(t)$ is measured $\forall t$, <u>without</u> noise

The initial state $x(t=1)$ is <u>unknown</u>

Goal: find a suitable estimate $\hat{x}(t)$, $\forall t$

Remark:

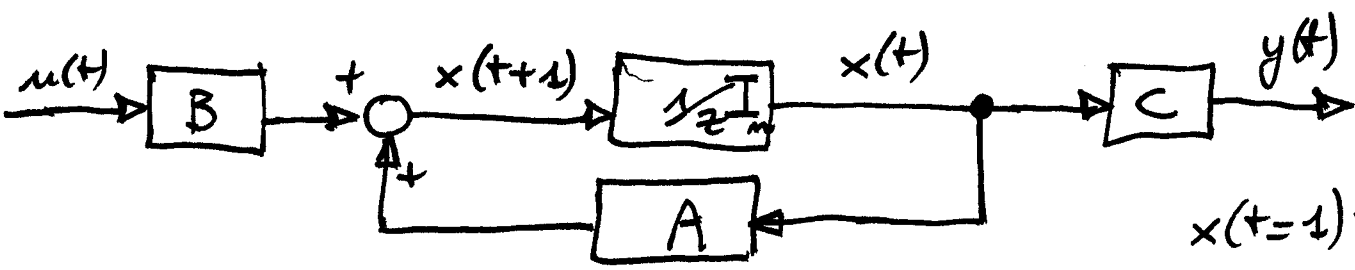
$$\mathcal{Z}\{x(t+1)\} = z\,\mathcal{Z}\{x(t)\} + z\,x(t=1)$$

$$\mathcal{Z}\{x(t-1)\} = \frac{1}{z}\,\mathcal{Z}\{x(t)\}$$

$$x(t) \longrightarrow \boxed{1/z} \longrightarrow x(t-1)$$

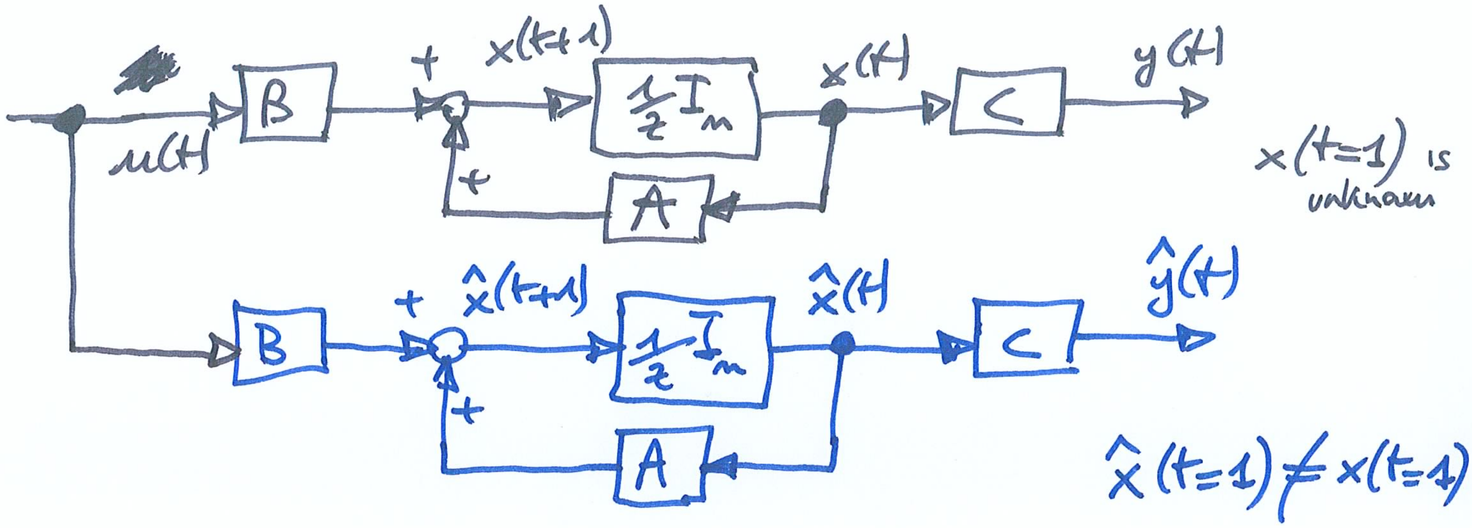
$$\mathcal{Z}\{x(t)\} = \frac{1}{z}\,\mathcal{Z}\{x(t+1)\}$$

$$x(t+1) \longrightarrow \boxed{\tfrac{1}{z}} \longrightarrow x(t)$$



$x(t=1)$: <u>unknown</u>

1) Solution #1: make a copy of the system, fed by the same input $u(t)$, with a different initial state



Let us define the state estimation error:

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

Let us consider the difference equation whose $\tilde{x}(t)$ is solution:

$$\boxed{\tilde{x}(t+1)} = \underline{x(t+1)} - \hat{x}(t+1) =$$

$$= A x(t) + \cancel{B u(t)} - [A\hat{x}(t) + \cancel{B u(t)}] =$$

$$= A x(t) - A\hat{x}(t) =$$

$$= A[x(t) - \hat{x}(t)] = \boxed{A\tilde{x}(t)} \Rightarrow$$

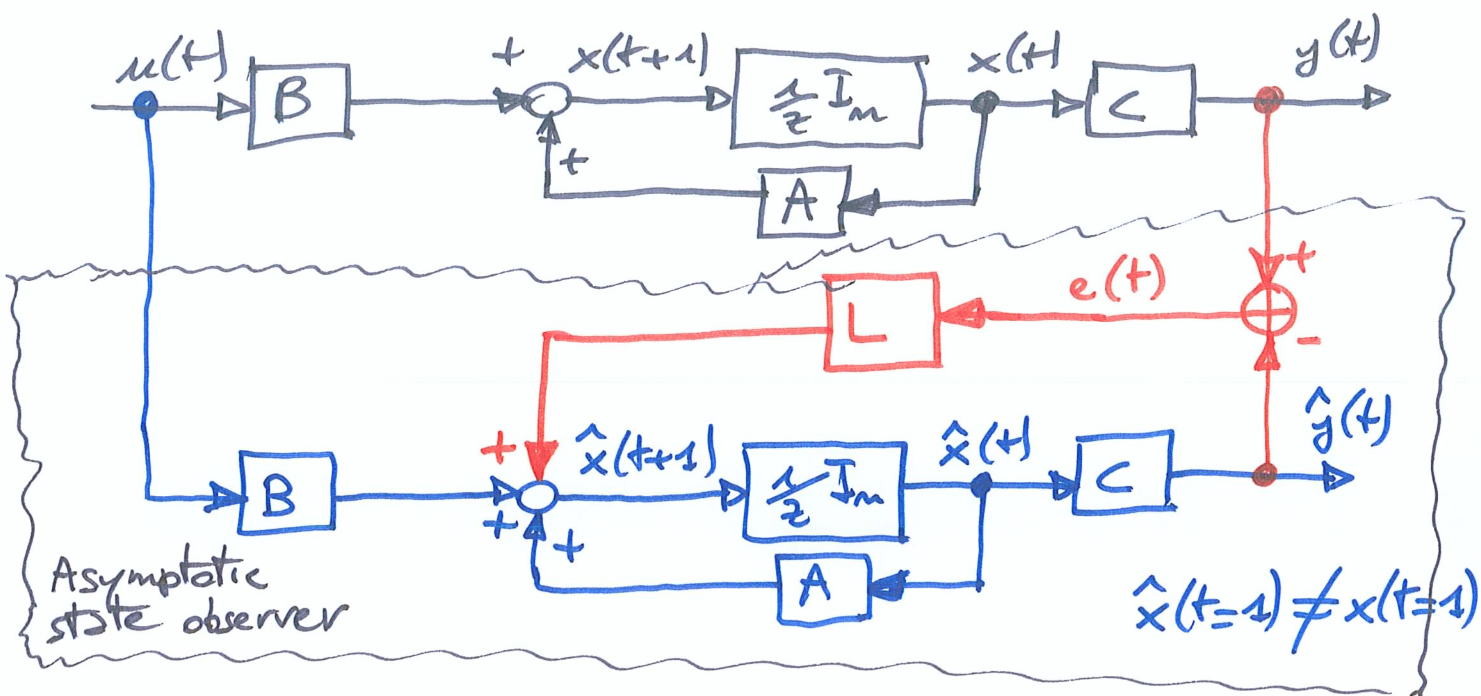
$$\tilde{x}(t) = A^{t-1}\tilde{x}(t=1)$$

IF $\lambda_i(A)$ are such $|\lambda_i(A)| < 1, \forall i \Rightarrow$

$\tilde{x}(t)$ converges to zero $\Rightarrow$

$\hat{x}(t)$ converges to $x(t)$ for $t \to \infty$

2) Solution # 2: modify the solution # 1 to exploit the information about $y(t)$



Let us consider the difference equation whose $\tilde{x}(t)$ is solution:

$$\boxed{\tilde{x}(t+1)} = x(t+1) - \hat{x}(t+1) =$$

$$= Ax(t) + \cancel{Bu(t)} - [A\hat{x}(t) + \cancel{Bu(t)} + L(y(t) - \hat{y}(t))] =$$

$$= A[x(t) - \hat{x}(t)] - L[Cx(t) - C\hat{x}(t)] =$$

$$= A\tilde{x}(t) - LC\tilde{x}(t) = \boxed{(A-LC)\tilde{x}(t)} \Rightarrow$$

$$\tilde{x}(t) = (A-LC)^{t-1}\tilde{x}(t=1)$$

IF $\lambda_i(A-LC)$ are such that $|\lambda_i(A-LC)| < 1, \forall i \Rightarrow$

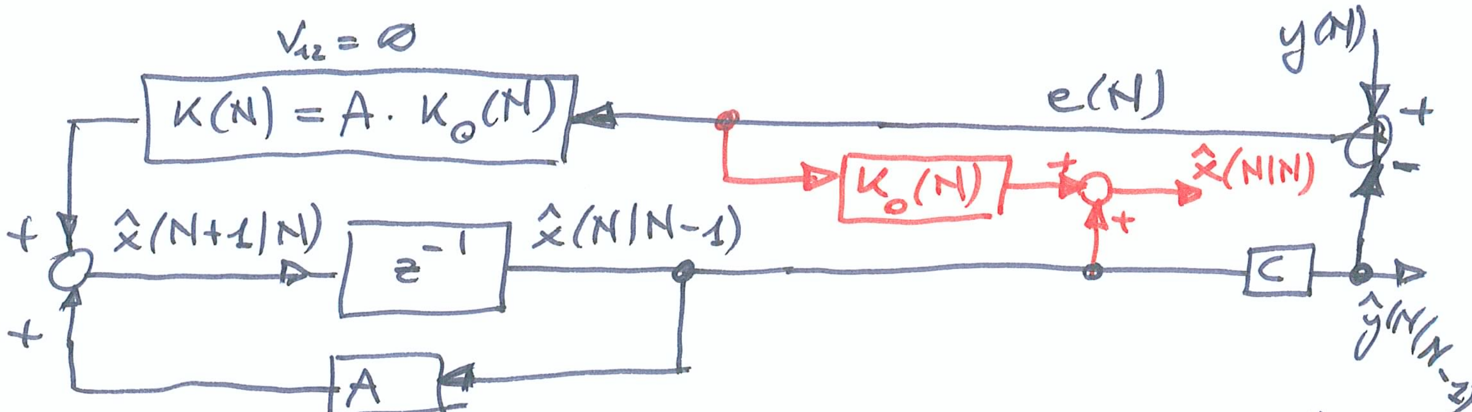
$\tilde{x}(t)$ converges to zero for $t \to \infty \Rightarrow$

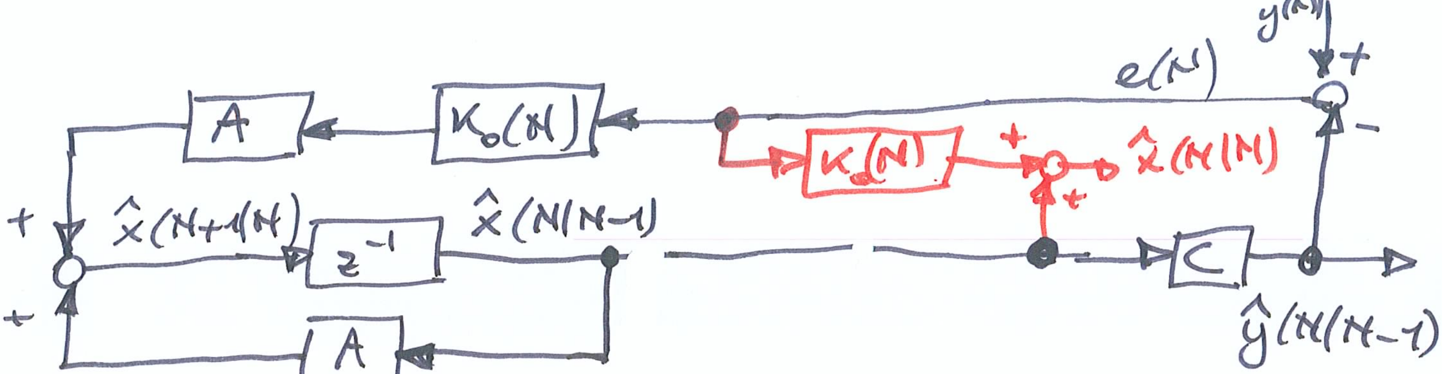
$\hat{x}(t)$ converges to $x(t)$ for $t \to \infty$ $\underline{\underline{\forall}}\, \tilde{x}(t=1)$

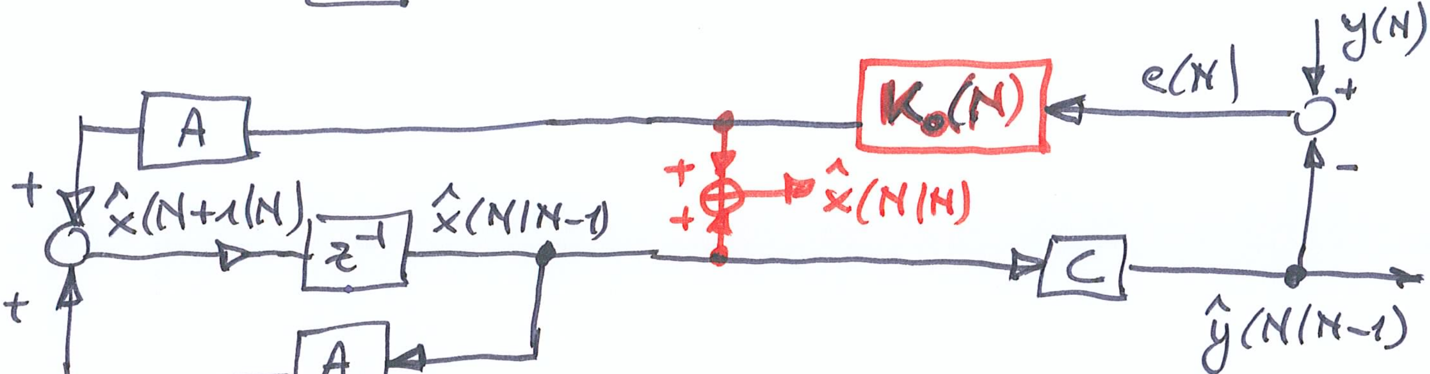
<u>Main theorem</u>: if the system is <u>fully observable</u>, i.e., $\rho(M_o) = n$, with $M_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$, then the matrix L can be designed such that $|\lambda_i(A-LC)| < 1, \ \forall i$

$$y^N = \begin{bmatrix} y(N) \\ \hline y(N-1) \\ y(N-2) \\ \vdots \\ y(1) \end{bmatrix} \qquad \begin{bmatrix} y(N-1) \\ \vdots \\ y(1) \end{bmatrix} = y^{N-1}$$

$$y^N = \begin{bmatrix} y(N) \\ \hline y^{N-1} \end{bmatrix}$$





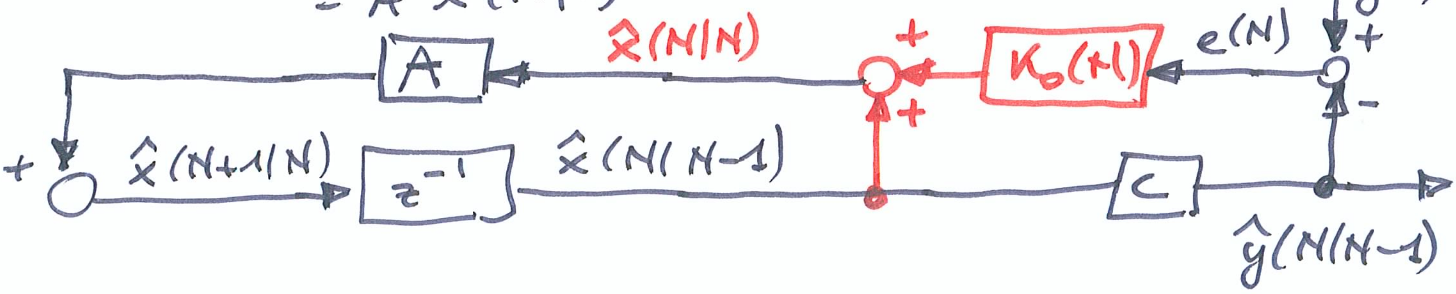


$$\hat{x}(N|N) = \hat{x}(N|N-1) + K_0(N)\, e(N)$$

$$\hat{x}(N+1|N) = A\, \hat{x}(N|N-1) + A\, K_0(N)\, e(N) =$$

$$= A\left[\hat{x}(N|N-1) + K_0(N)\, e(N)\right] =$$

$$= A\, \hat{x}(N|N)$$



LINEARIZATION OF NONLINEAR DYNAMIC SYSTEMS

Given a nonlinear dynamic system $\mathcal{S}$:

$$\mathcal{S}: \begin{cases} x(t+1) = f(t, x(t), u(t)) \\ y(t) = h(t, x(t), u(t)) \end{cases}$$

and a <u>nominal</u> movement of the system $\bar{x}(\cdot)$ obtained in correspondance to a <u>nominal</u> input $\bar{u}(\cdot)$, we can define the:

- state perturbation $\delta x(t) = x(t) - \bar{x}(t)$
- input perturbation $\delta u(t) = u(t) - \bar{u}(t)$
- output perturbation $\delta y(t) = y(t) - \bar{y}(t)$.

The perturbation dynamics are sufficiently well described by a <u>linearized dynamic system</u> $\mathcal{S}_\ell$, defined as:

$$\mathcal{S}_\ell: \begin{cases} \delta x(t+1) = A(t)\,\delta x(t) + B(t)\,\delta u(t) \\ \delta y(t) = C(t)\,\delta x(t) + D(t)\,\delta u(t) \end{cases}$$

[that is a linear, time-varying system], with:

$$A(t) = \text{Jacobian matrix of } f \text{ with respect to } x$$

$$= \left.\frac{\partial f}{\partial x}\right|_{\substack{x(t)=\bar{x}(t) \\ u(t)=\bar{u}(t)}}$$

$$B(t) = \text{Jacobian matrix of } f \text{ with respect to } u$$

$$= \left.\frac{\partial f}{\partial u}\right|_{\substack{x(t)=\bar{x}(t) \\ u(t)=\bar{u}(t)}}$$

$$C(t) = \text{Jacobian matrix of } h \text{ with respect to } x$$

$$= \left.\frac{\partial h}{\partial x}\right|_{\substack{x(t)=\bar{x}(t) \\ u(t)=\bar{u}(t)}}$$

$$D(t) = \left.\frac{\partial h}{\partial u}\right|_{\substack{x(t)=\bar{x}(t) \\ u(t)=\bar{u}(t)}}$$

Nonlinear system

$$x(t+1) = f(t, x(t)) + v_1(t)$$
$$y(t) = h(t, x(t)) + v_2(t)$$

$y(t)$

Nonlinear predictor

$$\hat{x}(N+1|N) = f(N, \hat{x}(N|N-1)) + \bar{K}(N)\, e(N)$$
$$\hat{y}(N+1|N) = h(N, \hat{x}(N|N-1))$$
$$e(N) = y(N) - \hat{y}(N|N-1)$$

$\hat{y}(N|N-1)$

Steady-state KALMAN predictor

Consider the system $\mathcal{S}$:

$$\mathcal{S}: \begin{cases} x(t+1) = Ax(t) + Bu(t) + v_1(t) & \text{state eq.} \\ y(t) = Cx(t) + Du(t) + v_2(t) & \text{output eq.} \end{cases}$$

with known matrices A, B, C, D
known exogenous input $u(t)$
$v_1(t), v_2(t)$: White Noise random variables

The steady-state Kalman predictor is

$$\mathcal{H}_\infty: \begin{cases} \hat{x}(N+1|N) = A\hat{x}(N|N-1) + Bu(N) + \bar{K}e(N) \\ \hat{y}(N|N-1) = C\hat{x}(N|N-1) + Du(N) \\ e(N) = y(N) - \hat{y}(N|N-1) \end{cases}$$

Note that:

$$\mathcal{H}_\infty: \begin{cases} \hat{x}(N+1|N) = A\hat{x}(N|N-1) + Bu(N) + \\ \qquad + \bar{K}\left[y(N) - C\hat{x}(N|N-1) - Du(N)\right] = \\ \qquad = (A - \bar{K}C)\hat{x}(N|N-1) + (B - \bar{K}D)u(N) + \\ \qquad + \bar{K}y(N) \\ \hat{y}(N|N-1) = C\hat{x}(N|N-1) + Du(N) \end{cases}$$

The internal stability of $\mathcal{H}_\infty$ depends on the matrix $\underline{\underline{A - \bar{K}C}}$

By defining: $\begin{bmatrix} u(N) \\ y(N) \end{bmatrix} = w(N) \Rightarrow$

$$\hat{x}(N+1|N) = \underbrace{(A - \bar{K}C)}_{A_{KE}}\hat{x}(N|N-1) + \underbrace{[B - \bar{K}D, \bar{K}]}_{B_{KE}}\begin{bmatrix} u(N) \\ y(N) \end{bmatrix}$$

$$\hat{y}(N|N-1) = \underbrace{C}_{C_{KE}}\hat{x}(N|N-1) + \underbrace{[D, \emptyset_{q\times q}]}_{D_{KE}}\begin{bmatrix} u(N) \\ y(N) \end{bmatrix}$$

$\Rightarrow$ the steady-state Kalman estimator is a LTI-system with matrices $\underbrace{A_{KE}, B_{KE}, C_{KE}, D_{KE}}_{\text{returned as I output parameter of kalman matlab command}}$

$\mathcal{H}_\infty$ under MATLAB

$$\mathcal{S}: \begin{cases} x[n+1] = Ax[n] + Bu[n] + Gw[n] , & \text{State eq.} \\ y[n] = Cx[n] + Du[n] + Hw[n] + v[n] , & \text{Output eq.} \end{cases}$$

Comparing with the previous setting:

$$w \longleftrightarrow v_1 , \qquad G = I_n , \qquad H = \emptyset_{q\times n} \qquad \Rightarrow$$
$$v \longleftrightarrow v_2$$

$$\mathcal{S}: \begin{cases} x[n+1] = Ax[n] + [B, \; G] \begin{bmatrix} u[n] \\ w[n] \end{bmatrix} = \\ \quad = \underbrace{A}_{A_\mathcal{S}} x[n] + \underbrace{[B, \; I_n]}_{B_\mathcal{S}} \begin{bmatrix} u[n] \\ w[n] \end{bmatrix} \\ y[n] = Cx[n] + [D, \; H] \begin{bmatrix} u[n] \\ w[n] \end{bmatrix} + v[n] = \\ \quad = \underbrace{C}_{C_\mathcal{S}} x[n] + \underbrace{[D, \; \emptyset_{q\times n}]}_{D_\mathcal{S}} \begin{bmatrix} u[n] \\ w[n] \end{bmatrix} + v[n] \end{cases}$$

With Matlab:

$$[\underbrace{KEST}_{ss(A_{KE}, B_{KE}, C_{KE}, D_{KE}, \pm 1)}, \; \underbrace{L}_{\overline{K}}, \; \underbrace{P}_{\overline{P}}, \; \underbrace{M}_{\overline{K}_o}, \; Z] = \text{kalman}(\underbrace{SYS}_{ss(A_\mathcal{S}, B_\mathcal{S}, C_\mathcal{S}, D_\mathcal{S}, \pm 1)}, \; \underbrace{QN}_{\Sigma_{v_1}}, \; \underbrace{RN}_{\Sigma_{v_2}}, \; \underbrace{NN}_{\Sigma_{v_1 v_2}})$$

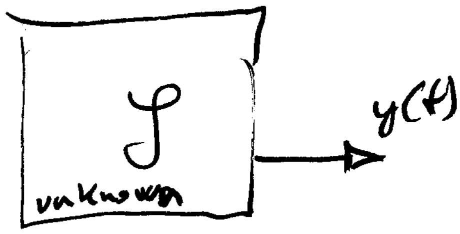
(Z: neglect!!)

with

$$\overline{K} = A \underbrace{\overline{P} C^T (C \overline{P} C^T + V_2)^{-1}}_{\overline{K}_o} = A \overline{K}_o$$

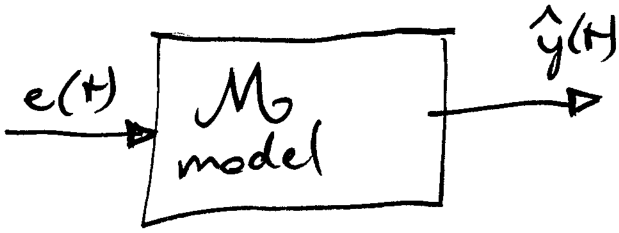
System $\mathcal{S}$ to be identified from data:

1) "time-series analysis":
we have only "output" measured data $y(t)$



$y(t)$: known

$\mathcal{S}$ can be modeled as:

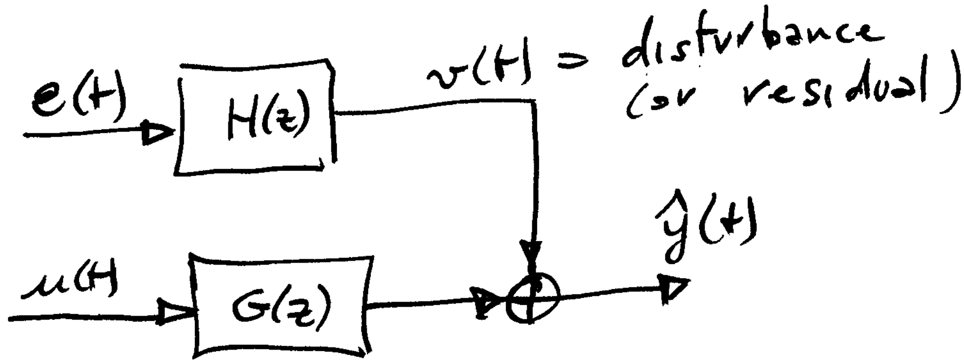


$e(t)$: "endogenous" input, for example $e(t) = \underline{\underline{WN(\emptyset, \Sigma_e)}}$

2) "input-output system"

$u(t), y(t)$: known

$\mathcal{S}$ can be modeled as:



$e(t) = WN(\emptyset, \Sigma_e)$

1) Equation error models:

In the time domain:

$$y(t) + a_1 y(t-1) + a_2 y(t-2) + \dots + a_{n_a} y(t-n_a) =$$
$$= b_1 u(t-1) + b_2 u(t-2) + \dots + b_{n_b} u(t-n_b) + e(t)$$

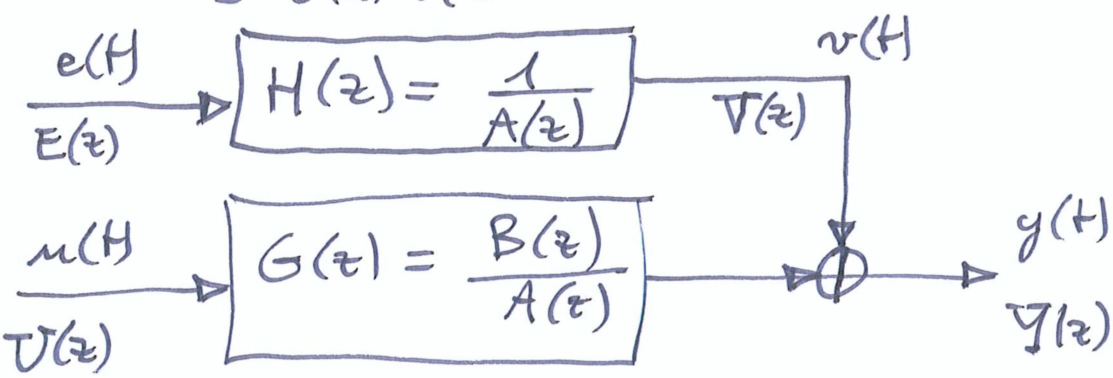
$$\Big\downarrow \mathcal{Z}$$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \dots + a_{n_a} z^{-n_a} Y(z) =$$
$$= b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \dots + b_{n_b} z^{-n_b} U(z) + E(z)$$

$$\underbrace{\left(1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_{n_a} z^{-n_a}\right)}_{A(z)} Y(z) =$$
$$= \underbrace{\left(b_1 z^{-1} + b_2 z^{-2} + \dots + b_{n_b} z^{-n_b}\right)}_{B(z)} U(z) + E(z)$$

$$A(z) Y(z) = B(z) U(z) + E(z)$$

$$Y(z) = \underbrace{\frac{B(z)}{A(z)}}_{G(z)} U(z) + \underbrace{\frac{1}{A(z)}}_{H(z)} E(z) =$$
$$= G(z) U(z) + H(z) E(z)$$



1.a) ARX model if $u(t)$ is present

$$\underbrace{A(z)\,Y(z)}_{\substack{\text{autoregressive}\\ \text{part}\\ (AR)}} = \underbrace{B(z)\,U(z)}_{\substack{\text{exogenous}\\ \text{part}\\ (X)}} + \underbrace{E(z)}_{WN}$$

If $n_a = 0 \Rightarrow A(z) = 1 + a_1 z^{-1} + \ldots + a_{n_a} z^{-n_a} = 1$

$$\Rightarrow A(z)Y(z) = Y(z) = B(z)\,U(z) + E(z)$$
$$= (b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{n_b} z^{-n_b})\,U(z) + E(z)$$

If $u(t) = \delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases} \Rightarrow$

$U(z) = 1 \Rightarrow$

$$Y(z) = B(z)U(z) + E(z) = B(z)\cdot 1 + E(z) =$$
$$= b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{n_b} z^{-n_b} + E(z) \xrightarrow{Z^{-1}}$$

$$y(t) = b_1\,\delta(t-1) + b_2\,\delta(t-2) + \ldots + b_{n_b}\,\delta(t-n_b) + e(t)$$

$\Rightarrow$ the impulse response $y(t)$ is finite

$\Rightarrow$ **FIR** model (Finite Impulse Response)

If $n_a \neq 0 \Rightarrow y(t)$ is not finite, in general

1.b) AR model if $u(t)$ is missing

$$\underbrace{A(z)\,Y(z)}_{\substack{\text{autoregressive}\\ \text{part}\\ (AR)}} = \underbrace{E(z)}_{WN}$$

2) ARMAX model

In the time-domain:

$$y(t) + a_1 y(t-1) + \dots + a_{n_a} y(t-n_a) =$$
$$= b_1 u(t-1) + \dots + b_{n_b} u(t-n_b) +$$
$$+ e(t) + c_1 e(t-1) + \dots + c_{n_c} e(t-n_c)$$

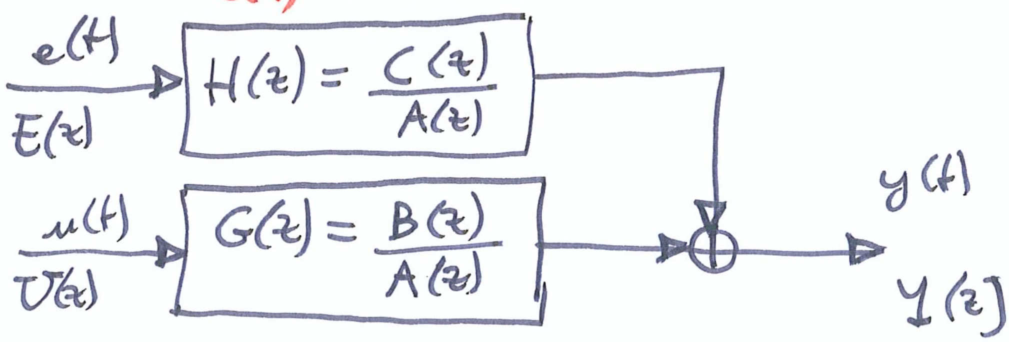
$$\Big\downarrow \mathcal{Z}$$

$$A(z)\, Y(z) = B(z)\, U(z) + C(z)\, E(z)$$

where

$$A(z) = 1 + a_1 z^{-1} + \dots + a_{n_a} z^{-n_a}$$
$$B(z) = b_1 z^{-1} + \dots + b_{n_b} z^{-n_b}$$
$$C(z) = 1 + c_1 z^{-1} + \dots + c_{n_c} z^{-n_c}$$

$$Y(z) = \underbrace{\frac{B(z)}{A(z)}}_{G(z)} U(z) + \underbrace{\frac{C(z)}{A(z)}}_{H(z)} E(z)$$



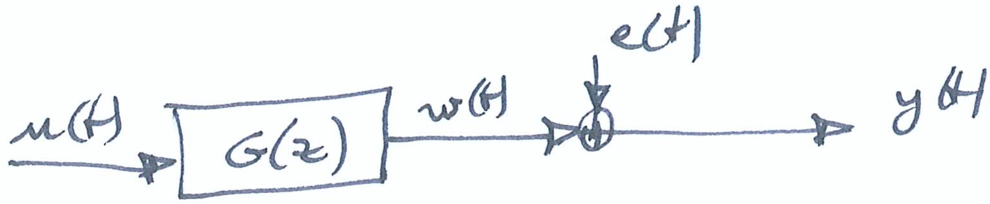
2.a) IF the input $u(t)$ is present:

$$\underbrace{A(z)\, Y(z)}_{\text{autoregressive part (AR)}} = \underbrace{B(z)\, U(z)}_{\text{exogenous part (X)}} + \underbrace{C(z)\, E(z)}_{\text{moving average part (MA)}}$$

2.b) IF the input $u(t)$ is missing:

$$\underbrace{A(z)\, Y(z)}_{\text{AR}} = \underbrace{C(z)\, E(z)}_{\text{MA}} \Rightarrow \underline{\text{ARMA model}}$$

3) Output Error model



where

$$w(t) + f_1 w(t-1) + f_2 w(t-2) + \dots + f_{n_F} w(t-n_F) =$$
$$= b_1 u(t-1) + b_2 u(t-2) + \dots + b_{n_b} u(t-n_b)$$

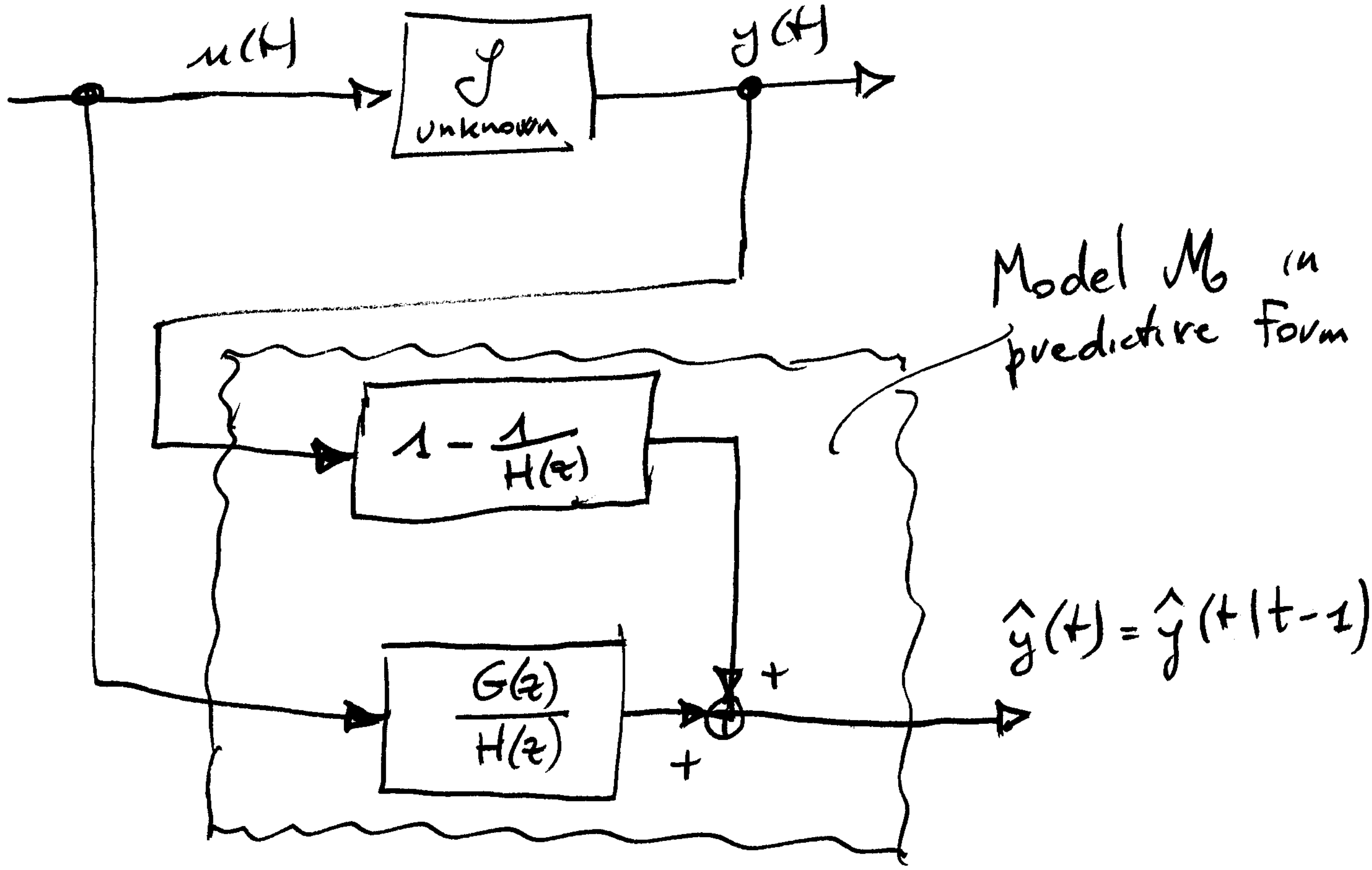
$$\Big\downarrow \mathcal{Z}$$

$$\underbrace{(1+f_1 z^{-1} + \dots + f_{n_F} z^{-n_F})}_{F(z)} W(z) = \overbrace{(b_1 z^{-1} + \dots b_{n_b} z^{-n_b})}^{B(z)} U(z)$$

$$F(z)\, W(z) = B(z)\, U(z) \Rightarrow$$
$$W(z) = \underbrace{\frac{B(z)}{F(z)}}_{G(z)} U(z) = G(z)\, U(z)$$

Since in the time-domain

$$y(t) = w(t) + e(t) \xrightarrow{\mathcal{Z}}$$
$$\boxed{Y(z) = W(z) + E(z) = G(z)\, U(z) + E(z)}$$


u(t)
$\mathcal{S}$
unknown
y(t)
Model $\mathcal{M}_6$ in predictive form
$1 - \frac{1}{H(z)}$
$\frac{G(z)}{H(z)}$
+
+
$\hat{y}(t) = \hat{y}(t|t-1)$

$$H(z) = \frac{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_m z^{-m}} =$$

$$= \frac{z^m + b_1 z^{m-1} + b_2 z^{m-2} + \ldots + b_m}{z^m + a_1 z^{m-1} + a_2 z^{m-2} + \ldots + a_m} = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots$$

$$\begin{array}{l|l}
z^m + b_1 z^{m-1} + b_2 z^{m-2} + \ldots + b_m & z^m + a_1 z^{m-1} + a_2 z^{m-2} + \ldots + a_m \\
\hline
-z^m - a_1 z^{m-1} - a_2 z^{m-2} - \ldots - a_m & 1 + \underbrace{(b_1 - a_1)}_{\alpha_1} z^{-1} + \alpha_2 z^{-2} + \ldots \\
\hline
(b_1 - a_1) z^{m-1} + (b_2 - a_2) z^{m-2} + \ldots & \\
\qquad + (b_m - a_m) & \\
-(b_1 - a_1) z^{m-1} - a_1 (b_1 - a_1) z^{m-2} - \ldots & \\
\qquad - a_m (b_1 - a_1) & \\
\hline
(b_2 - a_2 - a_1 b_1 + a_1^2) z^{m-2} + \ldots &
\end{array}$$

$$\frac{1}{H(z)} = \frac{z^m + a_1 z^{m-1} + a_2 z^{m-2} + \ldots + a_m}{z^m + b_1 z^{m-1} + b_2 z^{m-2} + \ldots + b_m} = 1 + \alpha_1' z^{-1} + \alpha_2' z^{-2} + \ldots$$

$$1 - \frac{1}{H(z)} = 1 - \left[1 + \alpha_1' z^{-1} + \alpha_2' z^{-2} + \ldots\right]$$

$$= -\alpha_1' z^{-1} - \alpha_2' z^{-2} - \ldots$$

$$\left[1 - \frac{1}{H(z)}\right] y(t) = \left[1 - \frac{1}{H(z)}\right] \mathcal{Z}\{y(t)\} = \left[-\alpha_1' z^{-1} - \alpha_2' z^{-2} - \ldots\right] \mathcal{Z}\{y(t)\} =$$

$$= -\alpha_1' z^{-1} \mathcal{Z}\{y(t)\} - \alpha_2' z^{-2} \mathcal{Z}\{y(t)\} - \ldots =$$

$$= -\alpha_1' \mathcal{Z}\{y(t-1)\} - \alpha_2' \mathcal{Z}\{y(t-2)\} \xrightarrow{\mathcal{Z}^{-1}}$$

$$-\alpha_1' y(t-1) - \alpha_2' y(t-2) - \ldots$$

$$\frac{1}{H(z)} = 1 + \alpha_1' z^{-1} + \alpha_2' z^{-2} + \dots$$

$$G(z) = \text{proper t.f.} = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}$$

$$\begin{array}{l|l}
b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n & z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n \\
\hline
-b_1 z^{n-1} - a_1 b_1 z^{n-2} - \dots - b_1 a_n z^{-1} & \underbrace{b_1}_{\gamma_1} z^{-1} + \underbrace{(b_2 - a_1 b_1)}_{\gamma_2} z^{-2} \\
\hline
(b_2 - a_1 b_1) z^{n-2} + \dots &
\end{array}$$

$$G(z) = \gamma_1 z^{-1} + \gamma_2 z^{-2} + \dots$$

$$G(z) \frac{1}{H(z)} = [\gamma_1 z^{-1} + \gamma_2 z^{-2} + \dots][1 + \alpha_1' z^{-1} + \alpha_2' z^{-2} + \dots] =$$

$$= \delta_1 z^{-1} + \delta_2 z^{-2} + \dots$$

$$\frac{G(z)}{H(z)} u(t) = \frac{G(z)}{H(z)} \mathcal{Z}\{u(t)\} = [\delta_1 z^{-1} + \delta_2 z^{-2} + \dots] \mathcal{Z}\{u(t)\} =$$

$$= \delta_1 z^{-1} \mathcal{Z}\{u(t)\} + \delta_2 z^{-2} \mathcal{Z}\{u(t)\} + \dots \xrightarrow{\mathcal{Z}^{-1}}$$

$$\delta_1 u(t-1) + \delta_2 u(t-2) + \dots$$

1) ARX model in predictive form

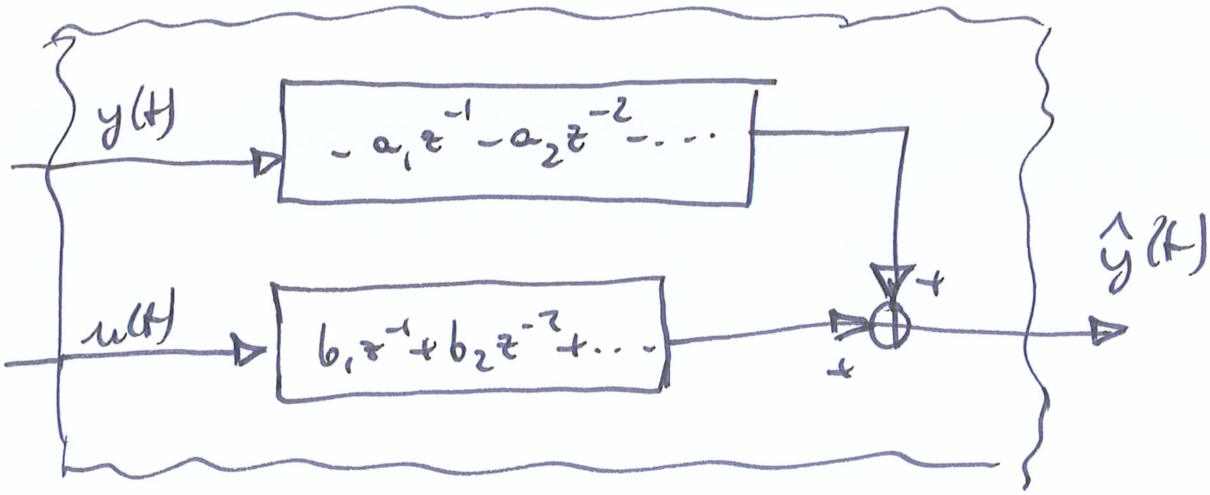
$$G(z) = \frac{B(z)}{A(z)}, \qquad H(z) = \frac{1}{A(z)} \Rightarrow$$

$$\hat{y}(t) = \hat{y}(t|t-1) = \left[1 - \frac{1}{H(z)}\right] y(t) + \frac{G(z)}{H(z)} u(t)$$

$$= [1 - A(z)]\, y(t) + \left[\frac{B(z)}{\cancel{A(z)}} \Big/ \frac{1}{\cancel{A(z)}}\right] u(t) =$$

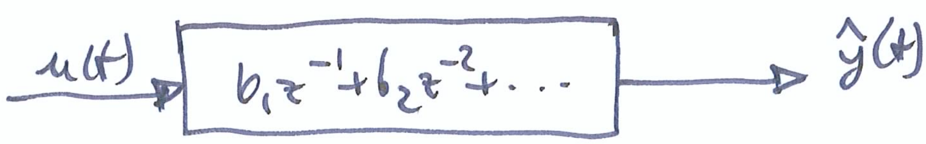
$$= [1 - \underbrace{A(z)}_{1 + a_1 z^{-1} + a_2 z^{-2} +}]\, y(t) + \underbrace{B(z)}_{= b_1 z^{-1} + b_2 z^{-2} + \dots} u(t) =$$

$$= [-a_1 z^{-1} - a_2 z^{-2} - \dots]\, y(t) + [b_1 z^{-1} + b_2 z^{-2} + \dots]\, u(t)$$



1.a) FIR model in predictive form

$$A(z) = 1 \Rightarrow [1 - A(z)]\, y(t) = \emptyset \Rightarrow$$



$$B(z) = b_1 z^{-1} + b_2 z^{-2} + \dots b_{n_b} z^{-n_b} =$$

$$= \frac{b_1}{z} + \frac{b_2}{z^2} + \dots + \frac{b_{n_b}}{z^{n_b}} =$$

$$= \frac{b_1 z^{n_b-1} + b_2 z^{n_b-2} + \dots + b_{n_b}}{z^{n_b}},$$

with n_b poles in $z=0$

$B(z)$ is asymptotically stable!!

$$1 - A(z) = -a_1 z^{-1} - a_2 z^{-2} - \dots - a_{n_a} z^{-n_a}$$

$$= -\frac{a_1}{z} - \frac{a_2}{z^2} - \dots - \frac{a_{n_a}}{z^{n_a}} =$$

$$= \frac{-a_1 z^{n_a-1} - a_2 z^{n_a-2} - \dots - a_{n_a}}{z^{n_a}},$$

with n_a poles in $z=0$

$1-A(z)$ is asymptotically stable!!

2) ARMAX model in predictive form

$$G(z) = \frac{B(z)}{A(z)}, \qquad H(z) = \frac{C(z)}{A(z)} \Rightarrow$$

$$\hat{y}(t) = \hat{y}(t|t-1) = \left[1 - \frac{1}{H(z)}\right] y(t) + \frac{G(z)}{H(z)} u(t) =$$

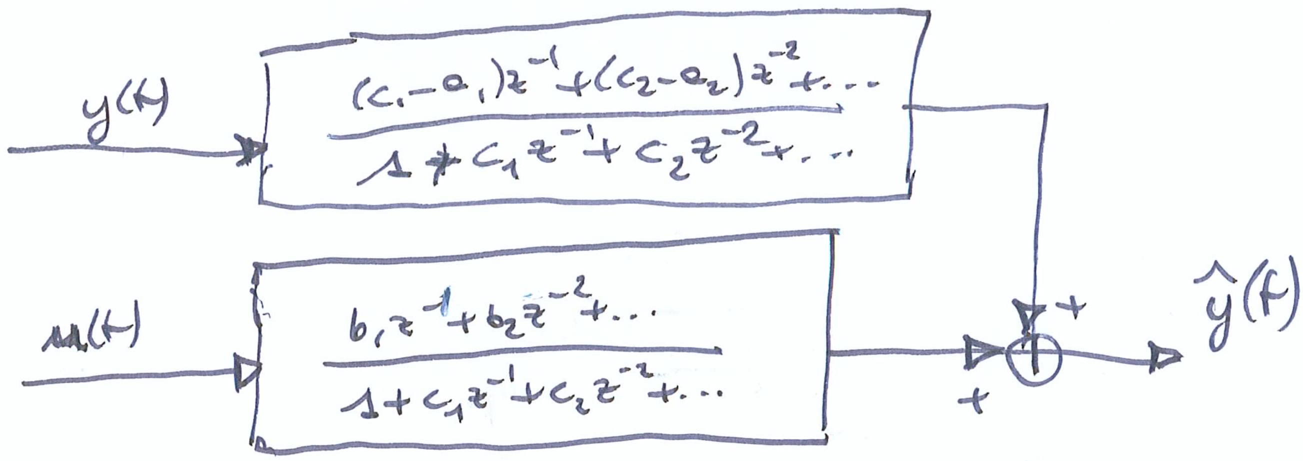
$$= \left[1 - \frac{A(z)}{C(z)}\right] y(t) + \left[\frac{B(z)}{\cancel{A(z)}} \Big/ \frac{C(z)}{\cancel{A(z)}}\right] u(t) =$$

$$= \left[1 - \frac{A(z)}{\underbrace{C(z)}_{1 + c_1 z^{-1} + c_2 z^{-2} + \dots}}\right] y(t) + \frac{B(z)}{C(z)} u(t)$$

$$= \left[\frac{\cancel{1} + c_1 z^{-1} + c_2 z^{-2} + \dots - (\cancel{1} + a_1 z^{-1} + a_2 z^{-2} + \dots)}{1 + c_1 z^{-1} + c_2 z^{-2} + \dots}\right] y(t) +$$

$$+ \frac{B(z)}{C(z)} u(t) =$$

$$= \frac{(c_1 - a_1) z^{-1} + (c_2 - a_2) z^{-2} + \dots}{1 + c_1 z^{-1} + c_2 z^{-2} + \dots} y(t) + \frac{b_1 z^{-1} + b_2 z^{-2} + \dots}{1 + c_1 z^{-1} + c_2 z^{-2} + \dots} u(t)$$



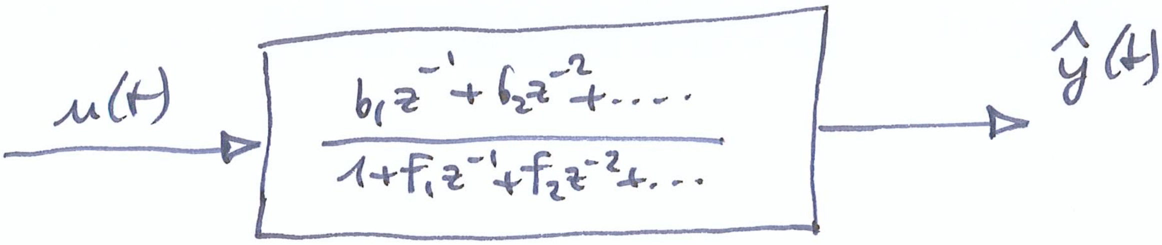
3) OE model in predictive Form

$$G(z) = \frac{B(z)}{F(z)}, \qquad H(z) = 1$$

$$\hat{y}(t) = \hat{y}(t|t-1) = \left[1 - \frac{1}{H(z)}\right] y(t) + \frac{G(z)}{H(z)} u(t) =$$

$$= [1-1]\, y(t) + \left[\frac{B(z)}{F(z)} \Big/ 1\right] u(t)$$

$$= \frac{B(z)}{\underbrace{F(z)}_{1 + f_1 z^{-1} + f_2 z^{-2} + \dots}} u(t) \qquad = \frac{b_1 z^{-1} + b_2 z^{-2} + \dots}{1 + f_1 z^{-1} + f_2 z^{-2} + \dots} u(t)$$



$\hat{y}(t)$ is independent of $y(t)$

$$\hat{y}(t) = \frac{B(z)}{F(z)} u(t)$$

$$F(z)\,\hat{y}(t) = B(z)\, u(t)$$

$$\varnothing = -F(z)\,\hat{y}(t) + B(z)\, u(t)$$

$$\hat{y}(t) = \hat{y}(t) - F(z)\,\hat{y}(t) + B(z)\, u(t)$$

$$= [1 - F(z)]\,\hat{y}(t) + B(z)\, u(t)$$

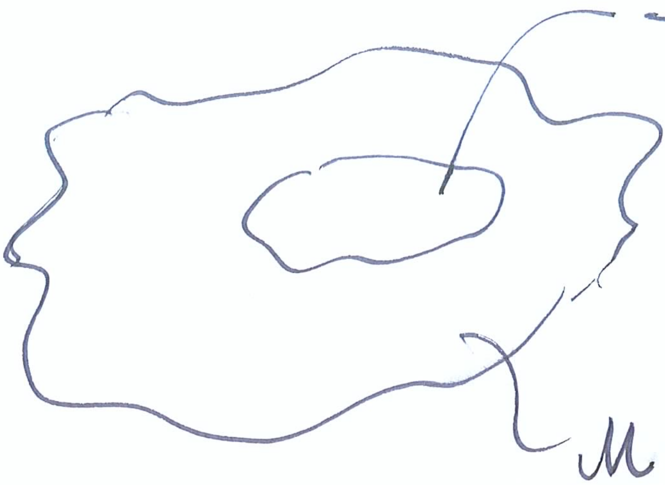
$$= [\cancel{1} - (\cancel{1} + f_1 z^{-1} + f_2 z^{-2} + \dots)]\,\hat{y}(t) + B(z)\, u(t)$$

$$= -(f_1 z^{-1} + f_2 z^{-2} + \dots)\,\hat{y}(t) + B(z)\, u(t)$$

$$= -f_1 z^{-1} \hat{y}(t) - f_2 z^{-2} \hat{y}(t) + \dots + B(z)\, u(t)$$

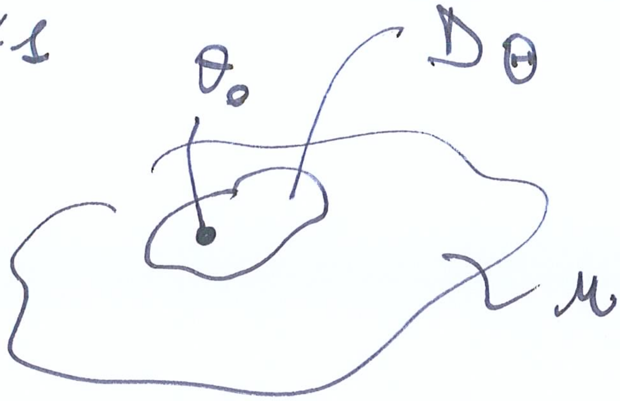
$$= -f_1\, \hat{y}(t-1) - f_2\, \hat{y}(t-2) + \dots + B(z)\, u(t)$$

$D_{\Theta} = \{ M : M$ is minimizing the cost function $J(\theta) \} \subset \mathcal{M}$



$\mathcal{M} = \{ M : M = M(\theta) \}$

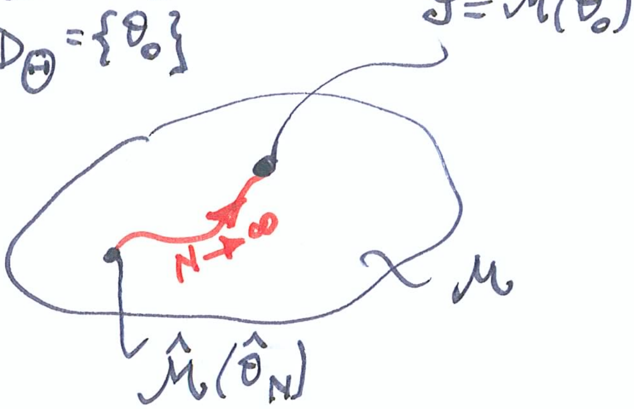
Result #1



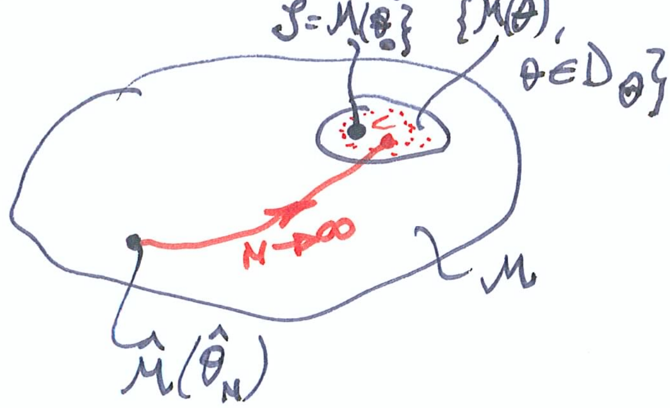
$\mathcal{S} \in \mathcal{M} \Rightarrow \theta_0 \in D_{\Theta}$

Result #2.a

$D_{\Theta} = \{\theta_0\}$



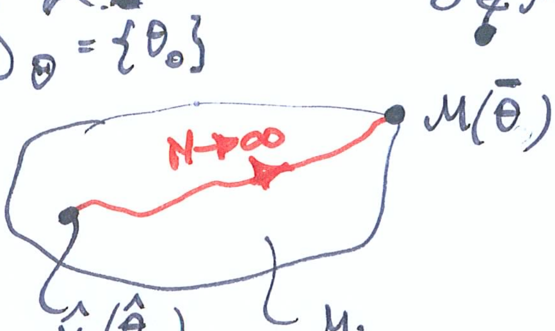
Result #2.b: $D_{\Theta} \supset \theta_0$



Result #2.c

$D_{\Theta} = \{\theta_0\}$

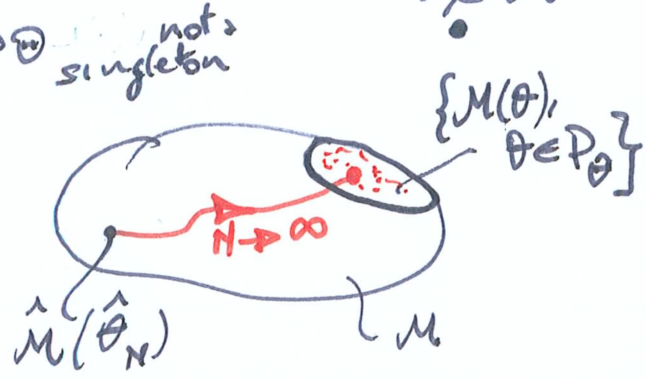
$\mathcal{S} \notin \mathcal{M}$



Result #2.d

D_{Θ} not a singleton

$\mathcal{S} \notin \mathcal{M}$



$$u(t) = \delta(t) = \begin{cases} 1 & t = 1 \\ 0 & t \neq 1 \end{cases}$$

$$r_u(\tau) = E\left[u(t)\, u(t-\tau)\right]$$

$$= E\left[\delta(t)\,\delta(t-\tau)\right] = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \left[\delta(t)\,\delta(t-\tau)\right]$$

$$= \begin{cases} \lim\limits_{N \to \infty} \frac{1}{N} \sum_{1}^{N} \left[\delta(t)\right]^2 = \lim\limits_{N \to \infty} \frac{1}{N} \cdot 1 = 0, & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$

$$u(t) = \varepsilon(t) = \begin{cases} 1 & t \geq 1 \\ 0 & t < 1 \end{cases}$$

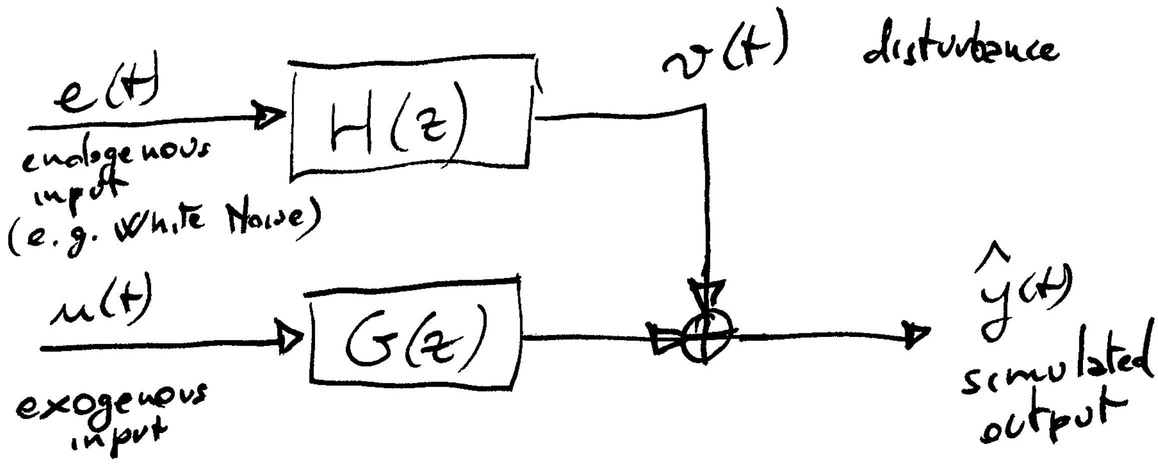
$$r_u(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \left[u(t)\, u(t-\tau)\right] =$$

$$= \begin{cases} \lim\limits_{N \to \infty} \frac{1}{N} \Big[\underbrace{1 \cdot 1 + \ldots 1 \cdot 1}_{N \text{ times}}\Big] = 1, & \tau = 0 \\ \lim\limits_{N \to \infty} \frac{1}{N} \Big[1 \cdot 0 + \underbrace{1 \cdot 1 + \ldots 1 \cdot 1}_{(N-\tau) \text{ times}}\Big] & \tau \neq 0 \end{cases}$$

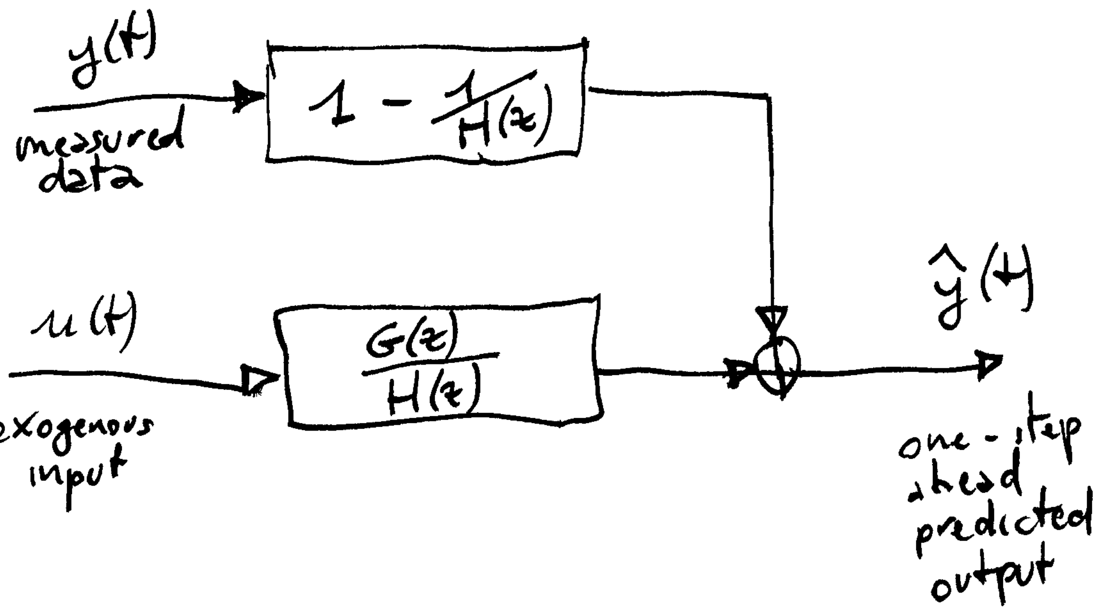
$$= \lim_{N \to \infty} \frac{N - \tau}{N} = 1$$

USE OF A MODEL FOR SIMULATION OR PREDICTION

1) SIMULATION

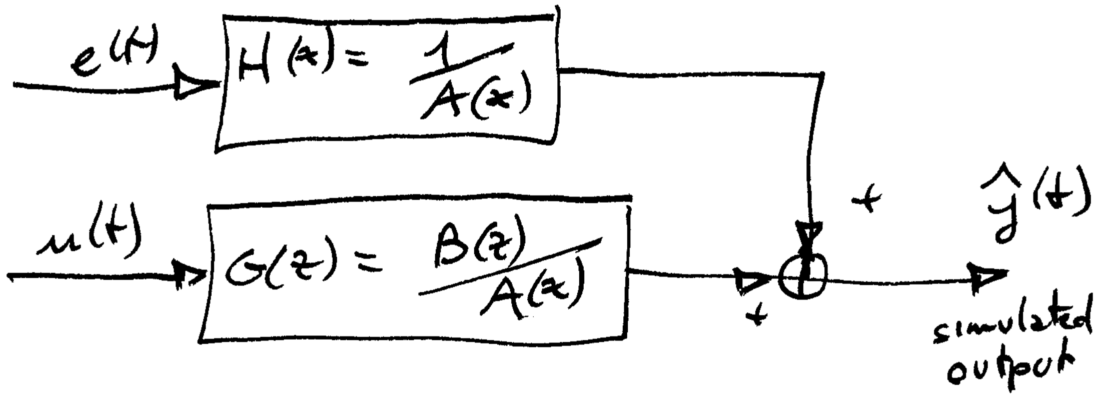


2) PREDICTION

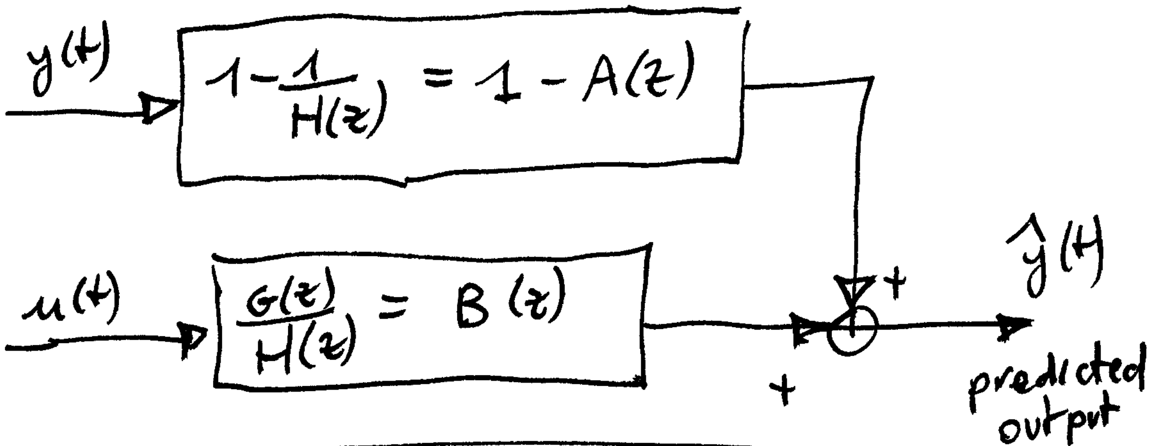


Case of ARX model: $G(z) = \frac{B(z)}{A(z)}$, $H(z) = \frac{1}{A(z)}$

1) ARX in simulation mode

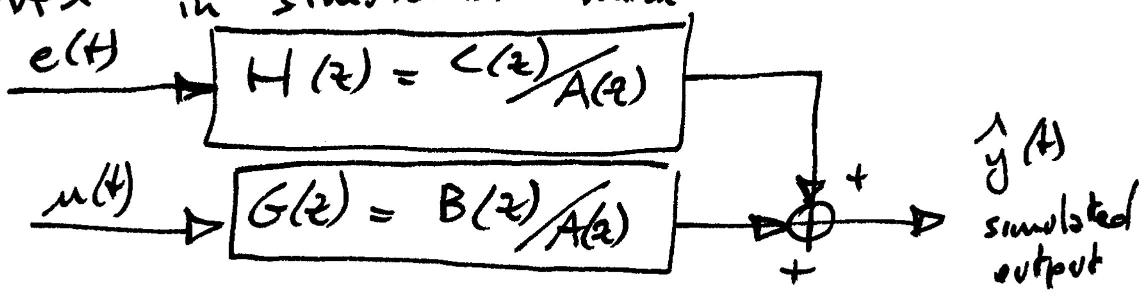


2) ARX in prediction mode

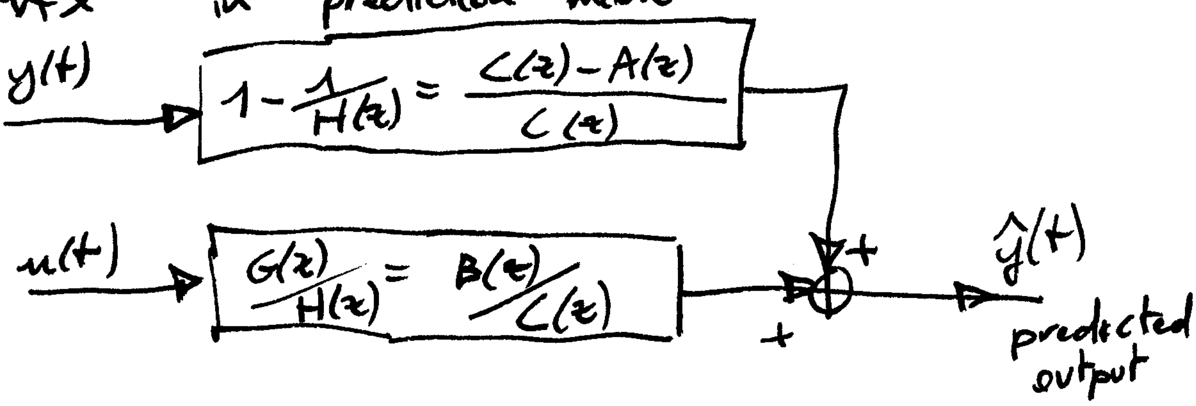


Case of ARMAX model: $G(z) = \frac{B(z)}{A(z)}$, $H(z) = \frac{C(z)}{A(z)}$

1) ARMAX in simulation mode

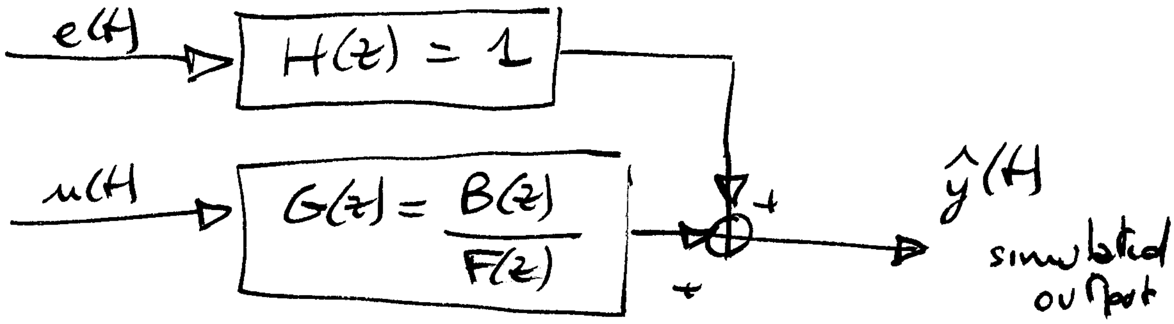


2) ARMAX in prediction mode

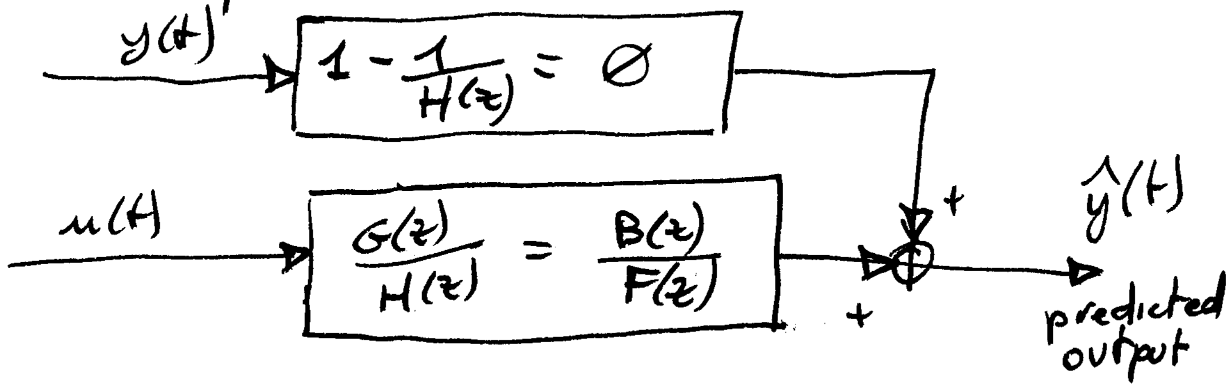


Case of OE model: $G(z) = \frac{B(z)}{F(z)}$, $H(z) = 1$

1) OE in simulation mode



2) OE in prediction mode



If $e(t) = \emptyset \Rightarrow \hat{y}(t)\Big|_{\text{simulated output}} \equiv \hat{y}(t)\Big|_{\text{predicted output}}$

```
% File Es4_arx

%Data generating system S: ARX(2,2) with a1_o=-1.2, a2_o=0.32, b1_o=1, b2_o=
0.5, e(.)=WGN(0,1), u(.)=WGN(0,4)

clear all, close all, pack

randn('state',0);

N=2000

sigma_u=4.0                   % input u(.) variance
u=sqrt(sigma_u)*randn(N,1);

sigma_e=1.0                   % noise e(.) variance
e=sqrt(sigma_e)*randn(N,1);

a1_o=-1.2, a2_o=0.32, b1_o=1, b2_o=0.5,
system_S=poly2th([1, a1_o, a2_o],[0, b1_o, b2_o]); % system S in theta format
y=idsim([u, e],system_S);    % data generation
Z=[y, u];

% Recursive estimate of ARX(2,2)

na=2;
nb=2;
max_na_nb=max([na,nb]);
PHI_y=toeplitz(-y(max_na_nb:N),-y(max_na_nb:-1:max_na_nb-na+1));
PHI_u=toeplitz( u(max_na_nb:N), u(max_na_nb:-1:max_na_nb-nb+1));
PHI=[PHI_y, PHI_u];

alpha=1e-0
V=alpha*eye(4);
theta=zeros(4,max_na_nb);
%t0=3
%V=inv(PHI(1:t0,:)'*PHI(1:t0,:))
%theta=[zeros(4,1), V*PHI(1:t0,:)'*y(1:t0)]
for t=max_na_nb+1:N,
    phi=PHI(t-max_na_nb,:)';
    beta=1+phi'*V*phi;
    V=V-(1/beta)*V*phi*phi'*V;
    e=y(t)-phi'*theta(:,t-1);
    K=V*phi;
    theta(:,t)=theta(:,t-1)+K*e;
end
theta(:,N)'
figure, plot(1:N,theta(1,:),[1,N],a1_o*[1,1],'r')
figure, plot(1:N,theta(2,:),[1,N],a2_o*[1,1],'r')
figure, plot(1:N,theta(3,:),[1,N],b1_o*[1,1],'r')
figure, plot(1:N,theta(4,:),[1,N],b2_o*[1,1],'r')
```