Rigid body dynamics

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Multiple point-mass bodies

Each mass is described by position \( \mathbf{r}_i = [x_i, y_i, z_i]^T \) and velocity \( \mathbf{v}_i = [v_{xi}, v_{yi}, v_{zi}]^T \), represented in a reference frame \( \mathcal{R}_b \).
Consider a pure rotation around a point $O$, for simplicity the origin of the reference frame $\mathcal{R}_b$.

If the angular velocity of the system is $\omega$, every mass will acquire a linear velocity $v_i$:

$$v_i(t) = \omega(t) \times r_i(t) = S(\omega(t))r_i(t)$$

The **linear momentum** $p_i(t)$ is defined as

$$p_i(t) = m_i v_i(t)$$

In the following slides the symbol $p$ indicates the linear momentum an not the cartesian pose or the point position.
Moment of a Force

Given a point mass located in a point $P$ represented by $\mathbf{r}$ and a force $\mathbf{f}$ applied to it, the moment of the force with respect to the point $O$ is given by

$$\mathbf{r} \times \mathbf{f} = \mathbf{r} \times m \frac{d\mathbf{v}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

Applying the derivative rules we obtain

$$\mathbf{r} \times \mathbf{f} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) - \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

$$= \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) - \mathbf{v} \times \mathbf{p}$$

where the last term is always zero, since $\mathbf{p}$ and the $\mathbf{v}$ are always collinear. The vector $\mathbf{h} = \mathbf{r} \times \mathbf{p}$ is called **angular momentum**.
Since $\mathbf{r} \times \mathbf{f}$ is often called **torque** produced by the force with respect to $O$, we can write

$$
\tau = \frac{d}{dt} \mathbf{h}
$$

Note the analogy

$$
\mathbf{f} = \frac{d}{dt} \mathbf{p} = \dot{\mathbf{p}} \quad \tau = \frac{d}{dt} \mathbf{h} = \dot{\mathbf{h}}
$$

We define **generalized momentum** $\mathbf{H}$ the vector that includes both momenta and **generalized force** the following

$$
\mathbf{F} = \begin{bmatrix} \mathbf{f} \\ \tau \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \mathbf{p} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{h}} \end{bmatrix} = \dot{\mathbf{H}}
$$
Angular momentum and inertia matrix

Given a point $O$ in space and a point mass $m_i$ with position $\mathbf{r}_i$, the angular momentum $\mathbf{h}_i$, is

$$\mathbf{h}_i(t) = \mathbf{r}_i(t) \times \mathbf{p}_i(t) = \mathbf{r}_i(t) \times (m_i \mathbf{v}_i(t))$$

Replacing $\mathbf{v}_i = \mathbf{\omega} \times \mathbf{r}_i$ and omitting for simplicity the time dependence, we obtain

$$\mathbf{h}_i = m_i (\mathbf{r}_i \times (\mathbf{\omega} \times \mathbf{r}_i))$$

Summing up all the $N$ point masses contributions, we have the total angular momentum

$$\mathbf{h} = \sum_{i=1}^{N} \mathbf{h}_i = \sum_{i} m_i (\mathbf{r}_i \times (\mathbf{\omega} \times \mathbf{r}_i))$$
Recalling the triple cross product property, we can write

\[ h = \sum_i m_i \left( (r_i^T r_i) \omega - (r_i^T \omega) r_i \right) \]

Replacing the square norm \( \|r_i\|^2 \) with the symbol \( r_i^2 \) and writing explicitly \( r_i^T \omega = (x_i \omega_x + y_i \omega_y + z_i \omega_z) \) we obtain

\[ h = \sum_i m_i \left( r_i^2 \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} - (x_i \omega_x + y_i \omega_y + z_i \omega_z) \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \right) \]

or

\[ h = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \sum_i \begin{bmatrix} m_i(r_i^2 - x_i^2) \omega_x - m_i x_i y_i \omega_y - m_i x_i z_i \omega_z \\ -m_i x_i y_i \omega_x + m_i(r_i^2 - y_i^2) \omega_y - m_i y_i z_i \omega_z \\ -m_i x_i z_i \omega_x - m_i y_i z_i \omega_y + m_i(r_i^2 - z_i^2) \omega_z \end{bmatrix} \]
The relation can be written in matrix form as

\[ \mathbf{h} = \sum_i \begin{bmatrix} \Gamma_{xx,i} & \Gamma_{xy,i} & \Gamma_{xz,i} \\ \Gamma_{yx,i} & \Gamma_{yy,i} & \Gamma_{yz,i} \\ \Gamma_{zx,i} & \Gamma_{zy,i} & \Gamma_{zz,i} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \sum_i \Gamma_i \omega \]

The new matrix \( \mathbf{\Gamma} \) is defined as

\[ \mathbf{\Gamma} = \begin{bmatrix} \Gamma_{xx} & \Gamma_{xy} & \Gamma_{xz} \\ \Gamma_{yx} & \Gamma_{yy} & \Gamma_{yz} \\ \Gamma_{zx} & \Gamma_{zy} & \Gamma_{zz} \end{bmatrix} = \sum_i \begin{bmatrix} \Gamma_{xx,i} & \Gamma_{xy,i} & \Gamma_{xz,i} \\ \Gamma_{yx,i} & \Gamma_{yy,i} & \Gamma_{yz,i} \\ \Gamma_{zx,i} & \Gamma_{zy,i} & \Gamma_{zz,i} \end{bmatrix} = \sum_i \Gamma_i \]

where

\[
\begin{align*}
\Gamma_{xx,i} &= m_i (r_i^2 - x_i^2) = m_i (y_i^2 + z_i^2) = m_i d_x^2 \\
\Gamma_{yy,i} &= m_i (r_i^2 - y_i^2) = m_i (x_i^2 + z_i^2) = m_i d_y^2 \\
\Gamma_{zz,i} &= m_i (r_i^2 - z_i^2) = m_i (x_i^2 + y_i^2) = m_i d_z^2
\end{align*}
\]

and

\[
\begin{align*}
\Gamma_{xy,i} &= \Gamma_{yx,i} = -m_i x_i y_i \\
\Gamma_{xz,i} &= \Gamma_{zx,i} = -m_i x_i z_i \\
\Gamma_{yz,i} &= \Gamma_{zy,i} = -m_i y_i z_i
\end{align*}
\]
Therefore

\[ h = \begin{bmatrix} \Gamma_{xx} & \Gamma_{xy} & \Gamma_{xz} \\ \Gamma_{yx} & \Gamma_{yy} & \Gamma_{yz} \\ \Gamma_{zx} & \Gamma_{zy} & \Gamma_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \Gamma \omega \]

or better, restating the time dependence

\[ h(t) = \Gamma(t)\omega(t) \]

The matrix \( \Gamma \) is called the **inertia matrix** or **inertia tensor** and has on the main diagonal the **inertia moments** defined as
Another possible representation of $h$ and $\Gamma$ can be obtained using the properties of the skew-symmetric matrices:

$$h = \sum_i m_i (r_i \times (\omega \times r_i)) = \sum_i m_i S(r_i) S(\omega) r_i$$

Recalling that $S(\omega)r_i = -S(r_i)\omega$, one obtains:

$$h = \sum_i m_i S(r_i) S(\omega) r_i = \sum_i -m_i S(r_i) S(r_i) \omega = \sum_i -m_i S^2(r_i) \omega$$

$$h = \left( -\sum_i m_i S^2(r_i) \right) \omega = \Gamma \omega$$

with

$$\Gamma = -\sum_i m_i S^2(r_i)$$
inertia matrix as:

\[ \mathbf{\Gamma} = \int_{\mathcal{V}} \rho(x, y, z) \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} \, dV \]

obtaining the inertia moments

\[ \Gamma_{xx} = \int_{\mathcal{V}} \rho(\mathbf{r})(y^2 + z^2) \, dV \]
\[ \Gamma_{yy} = \int_{\mathcal{V}} \rho(\mathbf{r})(x^2 + z^2) \, dV \]
\[ \Gamma_{zz} = \int_{\mathcal{V}} \rho(\mathbf{r})(x^2 + y^2) \, dV \]

and the inertia products

\[ \Gamma_{xy} = \Gamma_{yx} = - \int_{\mathcal{V}} \rho(\mathbf{r})xy \, dV \]
\[ \Gamma_{xz} = \Gamma_{zx} = - \int_{\mathcal{V}} \rho(\mathbf{r})xz \, dV \]
\[ \Gamma_{yz} = \Gamma_{zy} = - \int_{\mathcal{V}} \rho(\mathbf{r})yz \, dV \]
The inertia matrix is always defined specifying explicitly or implicitly a point with respect to which it is computed; usually this point is the center of mass of the body, and $R_b$ is the body frame.

If we consider another frame $R_k$, rotated around the common origin (in this case $C$) with respect to $R_b$, and the rotation matrix given by $R^k_b$, then the inertia matrix is

$$
\Gamma^k_c = R^k_b \Gamma^b_c R^b_k = R^k_b \Gamma^b_c (R^k_b)^T
$$

or, using the usual notation $R^a_b = (R^b_a)^T$

$$
\Gamma^k_c R^k_b = R^k_b \Gamma^b_c.
$$
Often it is appropriate to refer to a “privileged” reference frame $\mathcal{R}^*$, implicitly defined as that frame with its origin at the body center of mass and with respect to which the inertia matrix is diagonal, i.e.,

$$\Gamma^*_c = \begin{bmatrix} \Gamma_x & 0 & 0 \\ 0 & \Gamma_y & 0 \\ 0 & 0 & \Gamma_z \end{bmatrix};$$

This frame has the three unit vectors $\mathbf{i}^*$, $\mathbf{j}^*$ and $\mathbf{k}^*$, aligned along the so-called principal inertia axes; the matrix $\Gamma^*_c$ is called principal inertia matrix.
Parallel axes theorem

Assume that we know the inertia matrix $\Gamma$ with respect to a reference frame with origin in the center of mass $C$, and we want to compute a new inertia matrix $\Gamma'$ with respect to another frame, with different origin $O$, but with parallel axes.

$$
\Gamma' = \Gamma - mS^2(t)
$$

$$
= \Gamma + m \left[ ||t||^2 I - tt^T \right]
$$

$$
= \Gamma + m \begin{bmatrix}
(t_y^2 + t_z^2) & -t_x t_y & -t_x t_z \\
-t_x t_y & (t_x^2 + t_z^2) & -t_y t_z \\
-t_x t_z & -t_y t_z & (t_x^2 + t_y^2)
\end{bmatrix}
$$
Conclusions

**Generalized momentum** $\mathbf{H}(t)$

$$
\mathbf{H}(t) = \begin{bmatrix} p(t) \\ h(t) \end{bmatrix} = \begin{bmatrix} \mathbf{Mv}(t) & \text{linear momentum} \\ \Gamma(t)\omega(t) & \text{angular momentum} \end{bmatrix}
$$

**Generalized force** $\mathbf{F}(t)$

$$
\mathbf{F}(t) = \begin{bmatrix} f(t) \\ \tau(t) \end{bmatrix} = \dot{\mathbf{H}}(t) = \begin{bmatrix} \mathbf{M}\dot{v}(t) \\ \dot{\Gamma}(t)\omega(t) + \Gamma(t)\dot{\omega}(t) \end{bmatrix}
$$