FUNDAMENTALS OF
DYNAMIC SYSTEM IDENTIFICATION

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III Level Course 02LCPRV / 01LCPRV / 01LCPIU

“Experimental modeling: model building from experimental data”

Model building

- A **real system** $\mathcal{S}$ is studied for a certain **aim** or goal:
  - prediction/simulation
  - control
  - understanding/comprehension
  - design
  - diagnostics
- Two kinds of information are available:
  - **"a priori" info** : previous knowledge, plausible assumptions, physical laws, ...
  - **"a posteriori" info**: experimental measurements

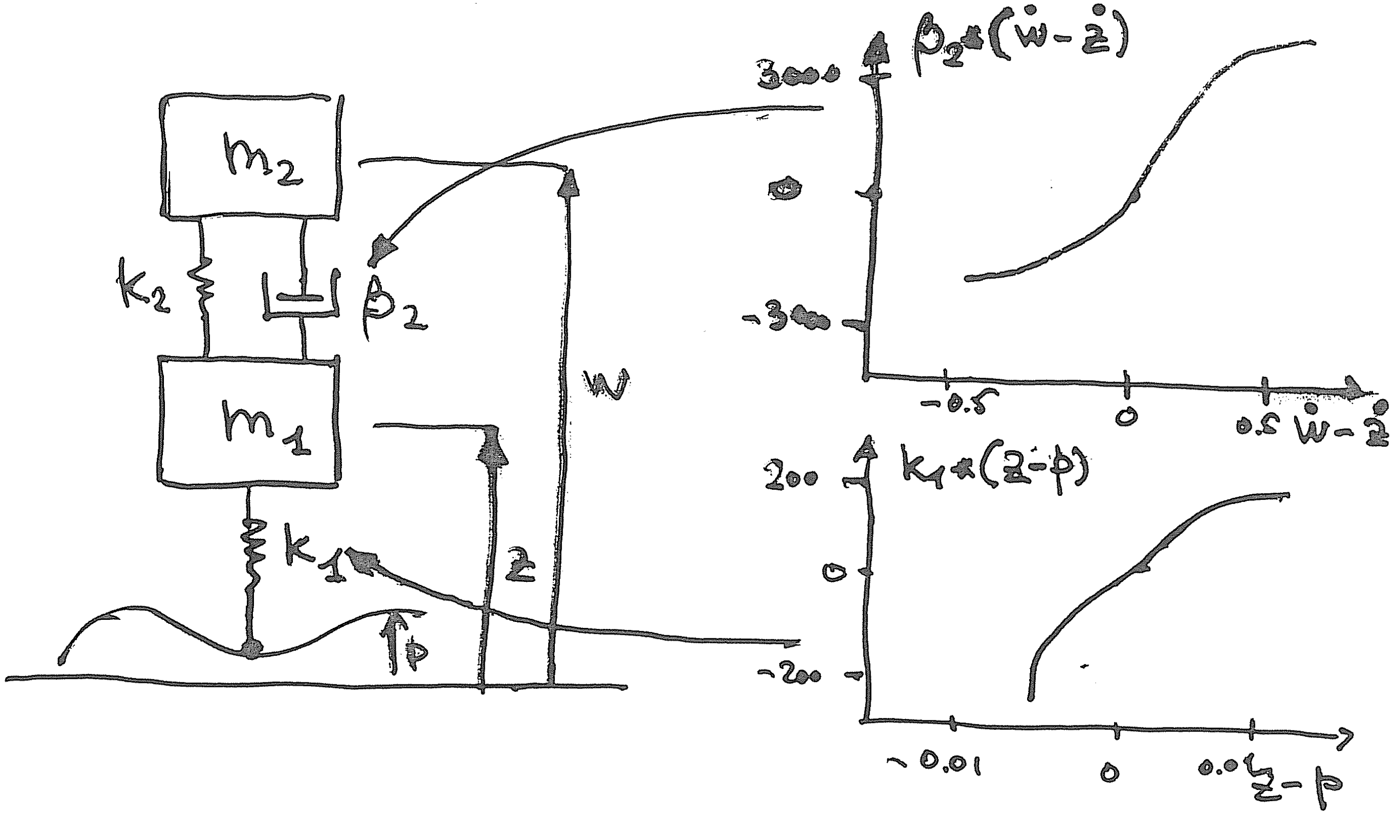
- “a priori” information $\Rightarrow$ theoretical structure of the mathematical model:

$$M(p), \quad \text{with } p : \text{unknown parameters}$$

- “a posteriori” information $\Rightarrow$ estimate of the parameters p
- Problem issues:
  - “consistency” evaluation of “a priori” and “a posteriori” informations
  - evaluation of estimation errors
- Typology of models:
  - **physical** (or “white box”) model: reproduction of the inner structure of $\mathcal{S}$
  - **black box** model: reproduction of the input-output behaviour of $\mathcal{S}$
  - **grey box** model: suitable combination of physical and black box model

Example #1: car suspension system

Assumption: the car chassis is rigid $\Rightarrow$ a "quarter car" model is used



m_2 : overall mass of the body of car with passengers, engine, etc.

k_2, β_2 : spring and damper representing the suspension system

m_1 : mass of the axis with the rigid part of the wheel

k_1 : spring representing the tyre

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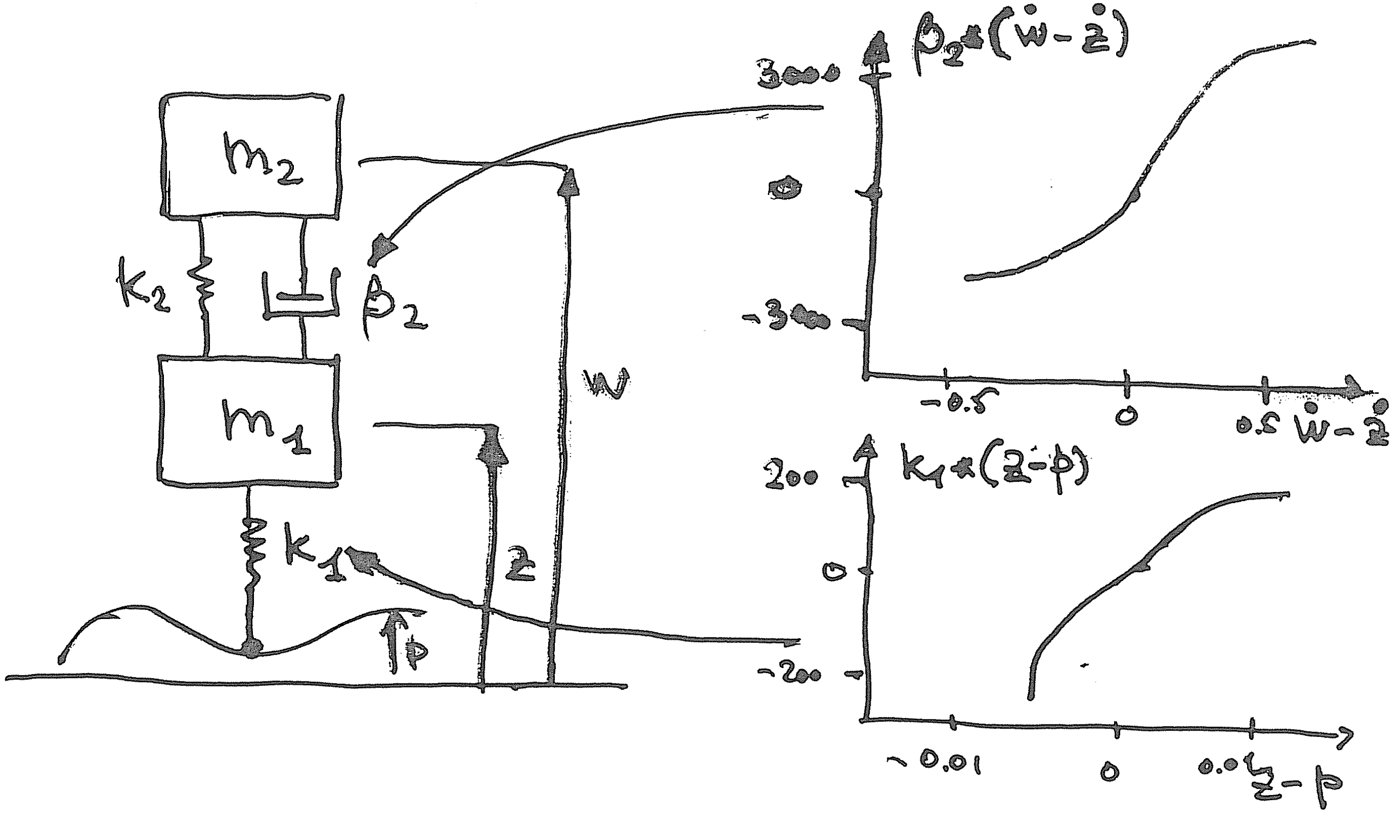

Goal: for low-frequency vertical movements, study the effects of the road profile p on:

- the body vertical acceleration $\ddot{w} = \frac{d^2 w}{dt^2}$ (to optimize the passenger comfort)
- the wheel vertical acceleration $\ddot{z} = \frac{d^2 z}{dt^2}$

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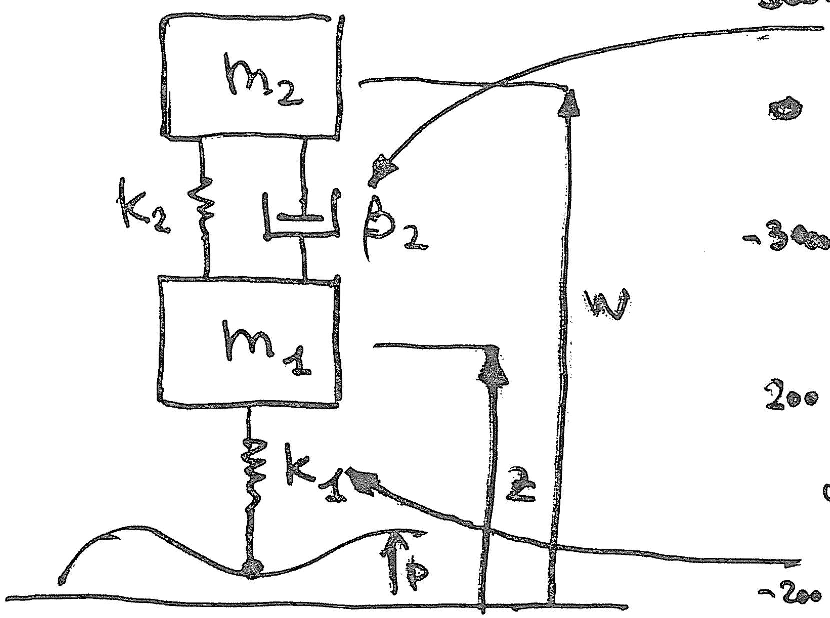

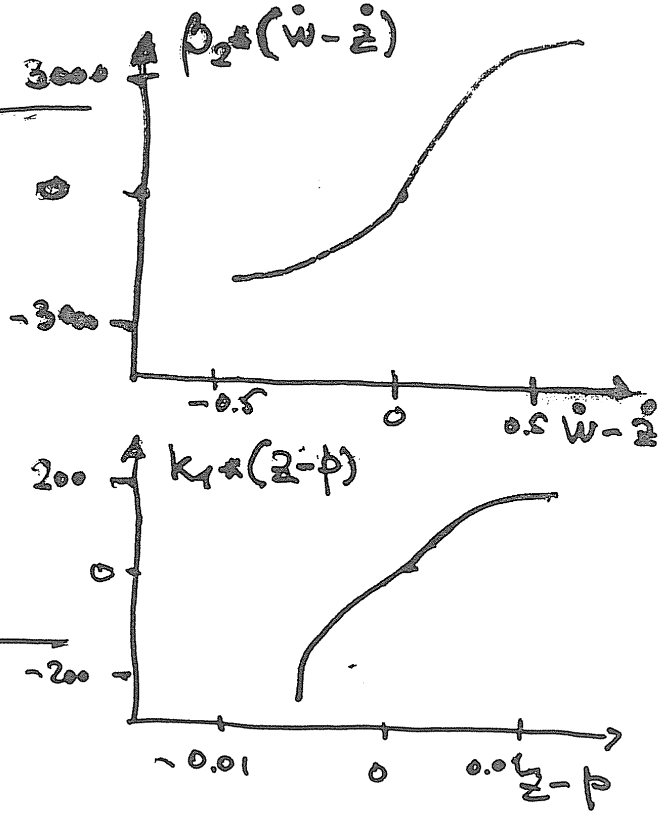

Further assumptions:

1) nonlinear characteristic of $\beta_2 \Rightarrow$ nonlinear characteristic between the body vs wheel relative speed $\dot{w} - \dot{z}$ and the damper force $\beta_2\,(\dot{w} - \dot{z})$ ($\dot{w} = \frac{dw}{dt}$: body vertical speed; $\dot{z} = \frac{dz}{dt}$: wheel vertical speed)

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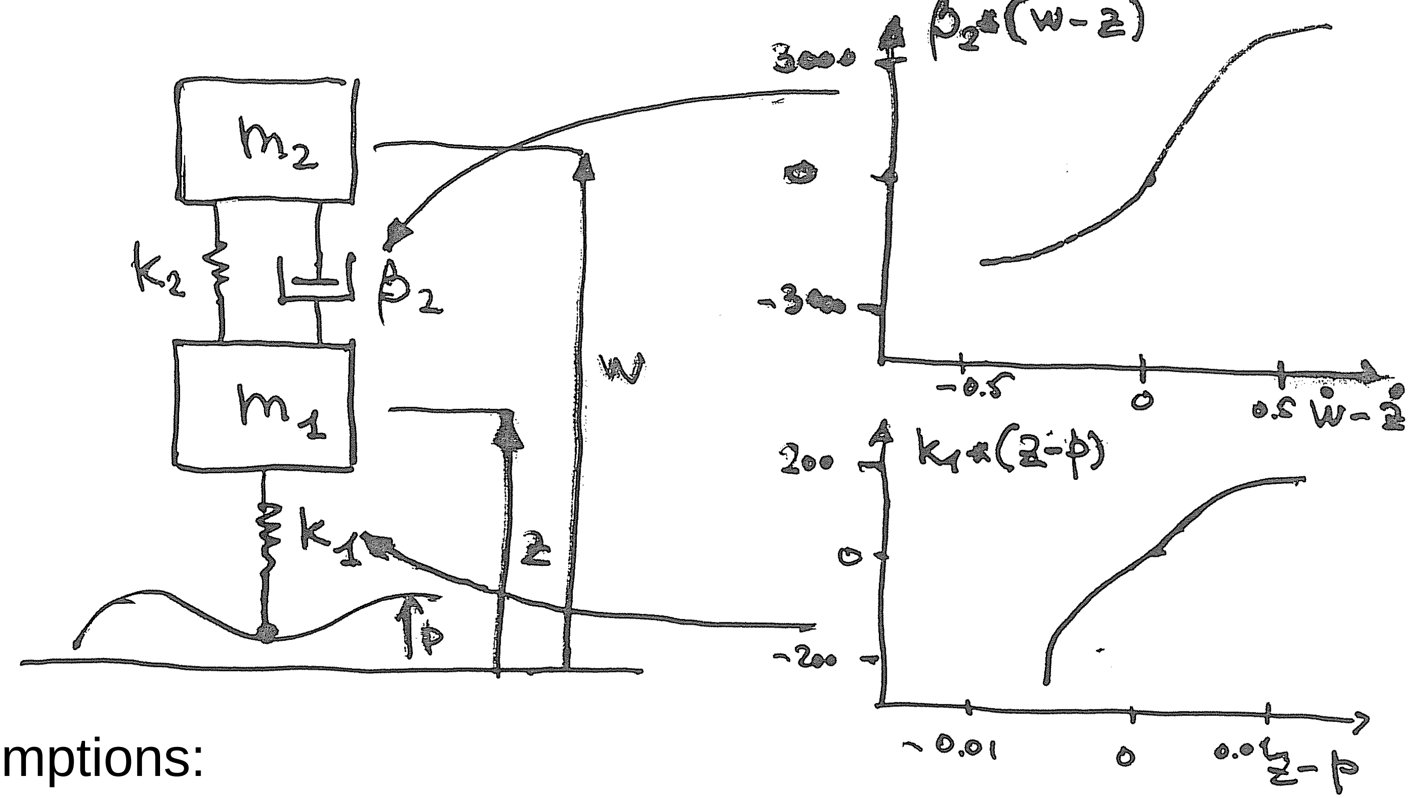
Assumption: the car chassis is rigid ⇒ a “quarter car” model is used



Further assumptions:

2) nonlinear characteristic of k_1 ⇒ nonlinear characteristic between the wheel vs road relative position $z-p$ and the elastic force $k_1\,(z-p)$ (z : wheel vertical position; p : road vertical position or road profile)

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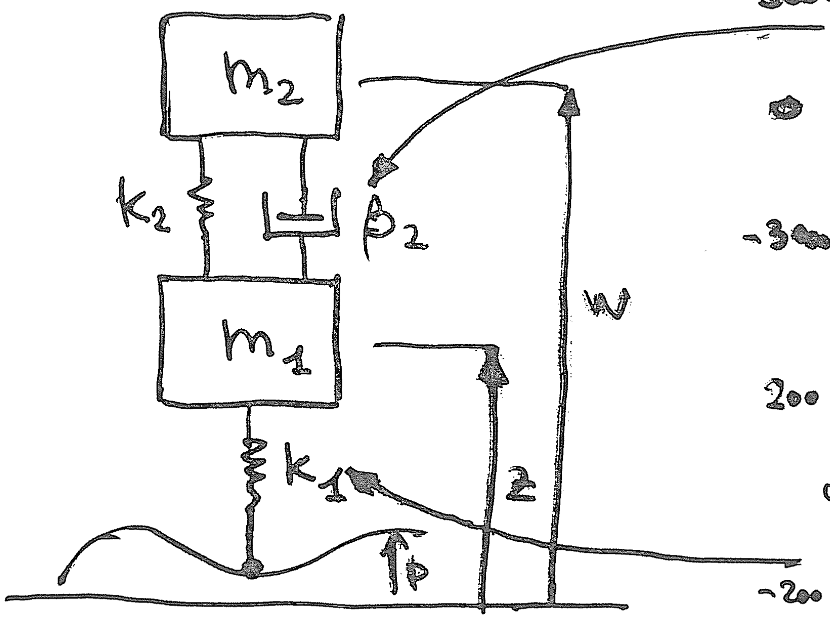
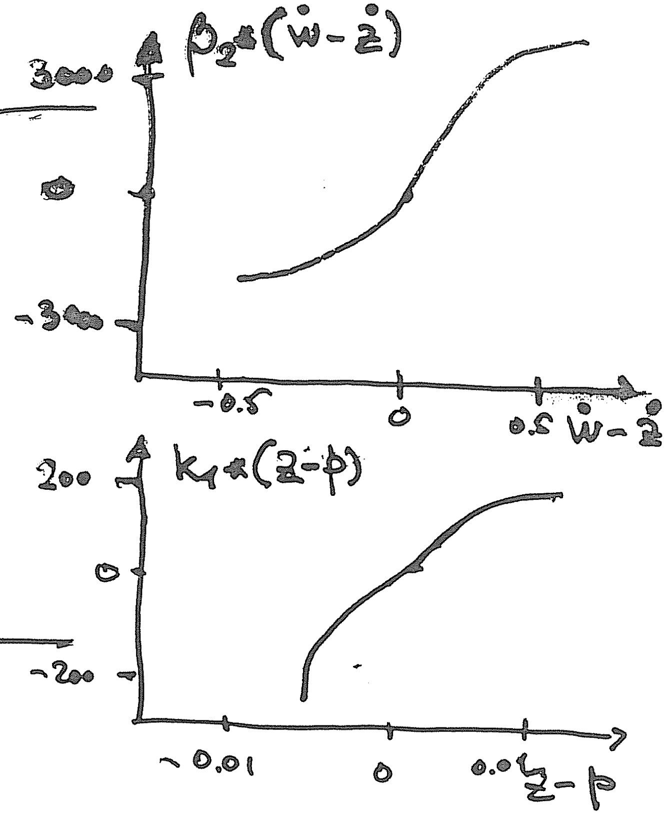
Assumption: the car chassis is rigid $\Rightarrow$ a "quarter car" model is used



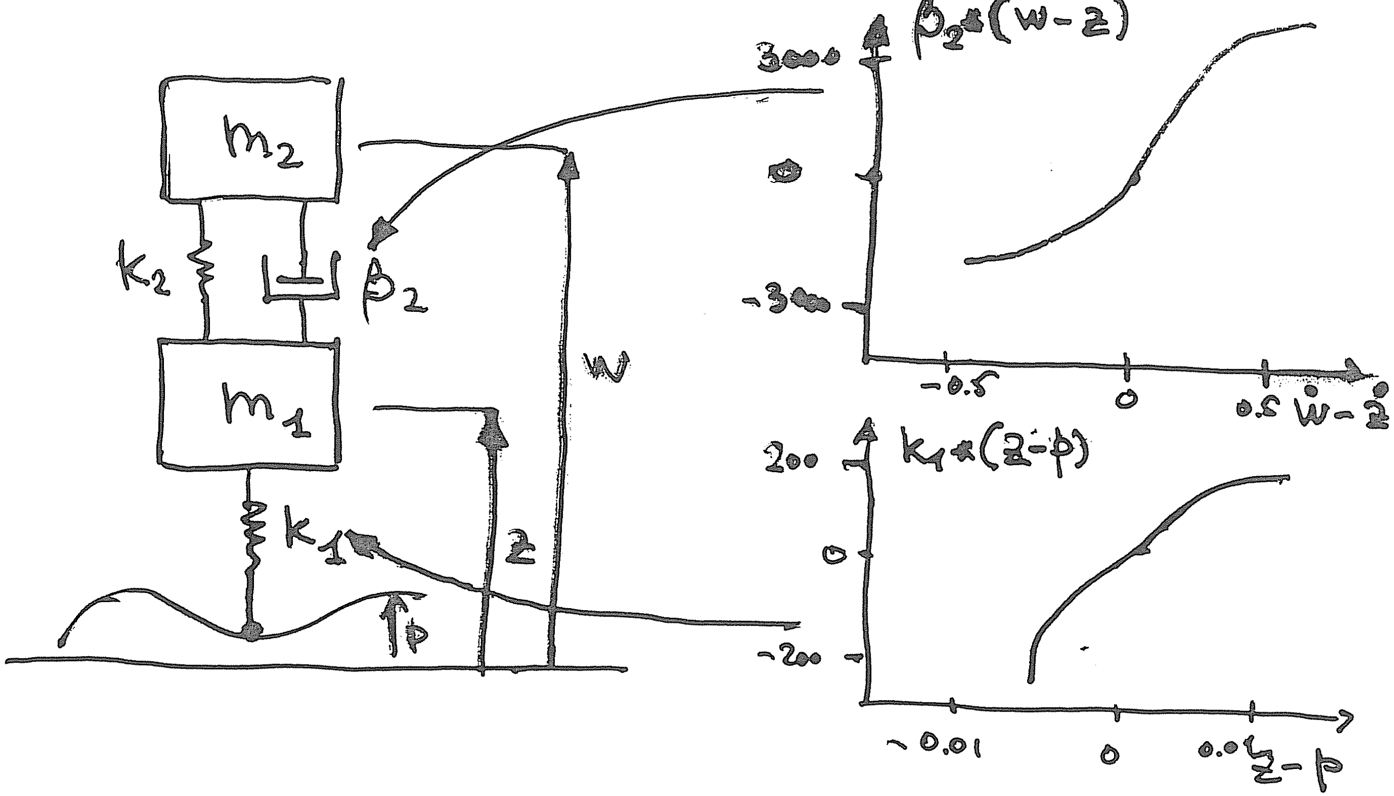


Further assumptions:

3) vertical weight forces $m_1 g$ and $m_2 g$ due to gravitational field are constant $\Rightarrow$ they are neglected, to focus only on the variations induced by the road profile

Example #1: car suspension system

Assumption: the car chassis is rigid $\Rightarrow$ a “quarter car” model is used



Using Newton’s laws of the vertical dynamics, a **continuous-time model** is derived:

$$\begin{cases} m_1\ddot{z} &= -\left[k_1\left(z-p\right)+k_2\left(z-w\right)+\beta_2\left(\dot{z}-\dot{w}\right)\right] \\ m_2\ddot{w} &= -\left[k_2\left(w-z\right)+\beta_2\left(\dot{w}-\dot{z}\right)\right] \end{cases}$$

A more general model is obtained by introducing the so-called **state variables**:

$$\begin{cases} x_1 = z \\ x_2 = \dot{z} \\ x_3 = w \\ x_4 = \dot{w} \end{cases}, \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$$\Downarrow$$

the two second-order differential equations of the physical model are transformed into four first-order differential equations:

$$\begin{cases} \dot{x}_1 = x_2 & = f_1(x(t), u(t)) \\ \dot{x}_2 = -\frac{k_1+k_2}{m_1}x_1 - \frac{\beta_2}{m_1}x_2 + \frac{k_2}{m_1}x_3 + \frac{\beta_2}{m_1}x_4 + \frac{k_1}{m_1}u & = f_2(x(t), u(t)) \\ \dot{x}_3 = x_4 & = f_3(x(t), u(t)) \\ \dot{x}_4 = \frac{k_2}{m_2}x_1 + \frac{\beta_2}{m_2}x_2 - \frac{k_2}{m_2}x_3 - \frac{\beta_2}{m_2}x_4 & = f_4(x(t), u(t)) \end{cases}$$

where $u(t) = p(t)$: **input variable** of the system $\Rightarrow$

a system of first-order differential equations called **state equations** is then derived:

$$\dot{x}(t) = f(x(t), u(t)) = \begin{bmatrix} f_1(x(t), u(t)) \\ f_2(x(t), u(t)) \\ f_3(x(t), u(t)) \\ f_4(x(t), u(t)) \end{bmatrix}$$

About the variables of interest (body and wheel accelerations):

$$\begin{cases} y_1 = \ddot{w} = \frac{k_2}{m_2}x_1 + \frac{\beta_2}{m_2}x_2 - \frac{k_2}{m_2}x_3 - \frac{\beta_2}{m_2}x_4 = \eta_1(x(t), u(t)) \\ y_2 = \ddot{z} = -\frac{k_1+k_2}{m_1}x_1 - \frac{\beta_2}{m_1}x_2 + \frac{k_2}{m_1}x_3 + \frac{\beta_2}{m_1}x_4 + \frac{k_1}{m_1}u = \eta_2(x(t), u(t)) \end{cases}$$

where $y(t)$: **output variables** of the system $\Rightarrow$

a system of static (or instantaneous) equations called **output equations** is derived:

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \eta(x(t), u(t)) = \begin{bmatrix} \eta_1(x(t), u(t)) \\ \eta_2(x(t), u(t)) \end{bmatrix}$$

- In general, state and output equations are nonlinear (with respect to x and u)
- If k_1, k_2, β_2 are constant (i.e., the system works in linearity conditions), while m_1 and m_2 are time-varying $\Rightarrow$ a **linear time-variant (LTV)** model is derived:

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t) \\ y(t) &= C(t)\,x(t) + D(t)\,u(t) \end{aligned}$$

where $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are suitable time-varying matrices:

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1(t)} & -\frac{\beta_2}{m_1(t)} & \frac{k_2}{m_1(t)} & \frac{\beta_2}{m_1(t)} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2(t)} & \frac{\beta_2}{m_2(t)} & -\frac{k_2}{m_2(t)} & -\frac{\beta_2}{m_2(t)} \end{bmatrix}, \qquad B(t) = \begin{bmatrix} 0 \\ \frac{k_1}{m_1(t)} \\ 0 \\ 0 \end{bmatrix}$$

$$C(t) = \begin{bmatrix} \frac{k_2}{m_2(t)} & \frac{\beta_2}{m_2(t)} & -\frac{k_2}{m_2(t)} & -\frac{\beta_2}{m_2(t)} \\ -\frac{k_1+k_2}{m_1(t)} & -\frac{\beta_2}{m_1(t)} & \frac{k_2}{m_1(t)} & \frac{\beta_2}{m_1(t)} \end{bmatrix}, \qquad D(t) = \begin{bmatrix} 0 \\ \frac{k_1}{m_1(t)} \end{bmatrix}$$

- If k_1, k_2, β_2, m_1, m_2 are constant (e.g., for short distances) $\Rightarrow$ a **linear time-invariant (LTI)** model is derived:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

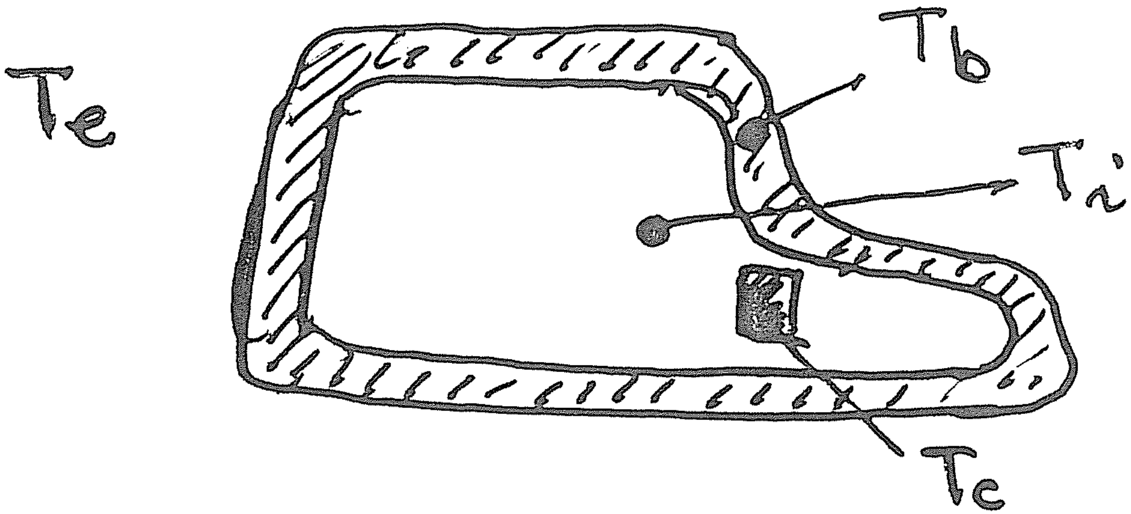
where A, B, C and D are suitable constant matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{\beta_2}{m_1} & \frac{k_2}{m_1} & \frac{\beta_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{\beta_2}{m_2} & -\frac{k_2}{m_2} & -\frac{\beta_2}{m_2} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{k_1}{m_1} \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{k_2}{m_2} & \frac{\beta_2}{m_2} & -\frac{k_2}{m_2} & -\frac{\beta_2}{m_2} \\ -\frac{k_1+k_2}{m_1} & -\frac{\beta_2}{m_1} & \frac{k_2}{m_1} & \frac{\beta_2}{m_1} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ \frac{k_1}{m_1} \end{bmatrix}$$

Example #2: car cooling system

Assumption: data are acquired ⇒ variables are know only at the sampling times ⇒ discrete-time variables must be considered



T_i : air temperature inside the car

T_e : external environment temperature

T_c : cooling system temperature

T_b : temperature of the body of the car

Goal: study the effects on T_i due to (small) time variations of T_e and T_c

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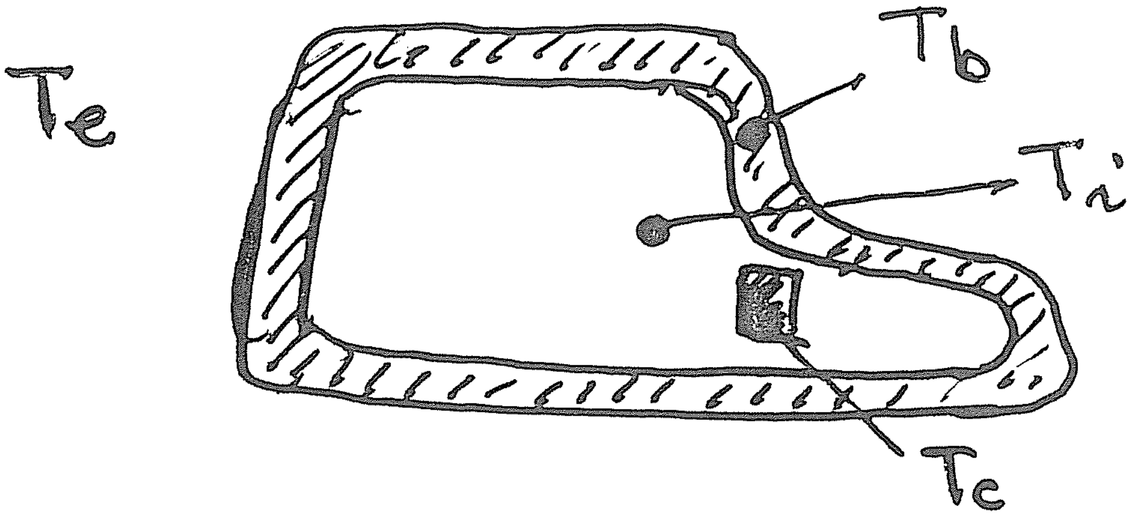

Further (simplifying) assumptions:

1) all the temperatures are not too much different

2) the air temperature T_i is constant inside the car

3) the temperature gradients of T_e, T_c and T_b are neglected

Example #2: car cooling system

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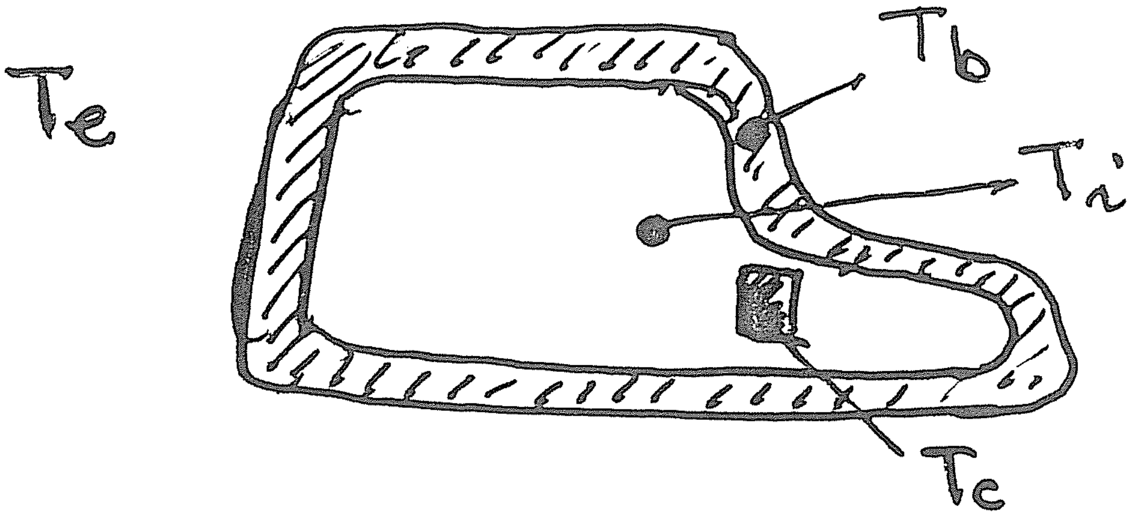

Considering all the thermal exchanges, a **discrete-time model** is derived:

$$\begin{cases} c_i m_i \left[T_i\left((j+1)\Delta t\right) - T_i\left(j\Delta t\right)\right] = \\ \qquad = -\left\{k_{ic}\left[T_i\left(j\Delta t\right) - T_c\left(j\Delta t\right)\right]^m + k_{ib}\left[T_i\left(j\Delta t\right) - T_b\left(j\Delta t\right)\right]\right\} \\ c_b m_b \left[T_b\left((j+1)\Delta t\right) - T_b\left(j\Delta t\right)\right] = \\ \qquad = -\left\{k_{ib}\left[T_b\left(j\Delta t\right) - T_i\left(j\Delta t\right)\right] + k_{be}\left[T_b\left(j\Delta t\right) - T_e\left(j\Delta t\right)\right]\right\} \end{cases}$$

Considering all the thermal exchanges, a **discrete-time model** is derived:

$$\begin{cases} c_i m_i \left[T_i\left((j+1)\,\Delta t\right) - T_i\left(j\Delta t\right)\right] = \\ \qquad = -\left\{k_{ic}\left[T_i\left(j\Delta t\right) - T_c\left(j\Delta t\right)\right]^m + k_{ib}\left[T_i\left(j\Delta t\right) - T_b\left(j\Delta t\right)\right]\right\} \\ c_b m_b \left[T_b\left((j+1)\,\Delta t\right) - T_b\left(j\Delta t\right)\right] = \\ \qquad = -\left\{k_{ib}\left[T_b\left(j\Delta t\right) - T_i\left(j\Delta t\right)\right] + k_{be}\left[T_b\left(j\Delta t\right) - T_e\left(j\Delta t\right)\right]\right\} \end{cases}$$

Δt : sampling time

c_i : thermal capacity of the air inside the car (almost constant at 20^o)

m_i : mass of the air inside the car

k_{ic} : thermal conductivity between the air inside the car and the cooling system

$m \in [1, 1.5]$: parameter depending on the kind of cooling system and on how it is placed

k_{ib} : thermal conductivity between the air inside the car and the body of the car

c_b : thermal capacity of the body of the car

m_b : mass of the body of the car

k_{be} : thermal conductivity between the body of the car and the external environment

A more general model is obtained by introducing the so-called **state variables**:

$$\begin{cases} x_1(j) = T_i(j\Delta t) \\ x_2(j) = T_b(j\Delta t) \end{cases}, \qquad x(j) = \begin{bmatrix} x_1(j) \\ x_2(j) \end{bmatrix}$$

and the **input variables**:

$$\begin{cases} u_1(j) = T_c(j\Delta t) \\ u_2(j) = T_e(j\Delta t) \end{cases}, \qquad u(j) = \begin{bmatrix} u_1(j) \\ u_2(j) \end{bmatrix}$$

$$\Downarrow$$

two first-order difference equations are obtained:

$$\begin{cases} x_1(j+1) = x_1(j) - \frac{1}{c_i m_i}\{k_{ic}[x_1(j) - u_1(j)]^m + k_{ib}[x_1(j) - x_2(j)]\} = \\ \qquad = \frac{c_i m_i - k_{ib}}{c_i m_i} x_1(j) + \frac{k_{ib}}{c_i m_i} x_2(j) - \frac{k_{ic}}{c_i m_i}[x_1(j) - u_1(j)]^m = f_1(x(j), u(j)) \\ x_2(j+1) = x_2(j) - \frac{1}{c_b m_b}\{k_{ib}[x_2(j) - x_1(j)] + k_{be}[x_2(j) - u_2(j)]\} = \\ \qquad = \frac{k_{ib}}{c_b m_b} x_1(j) + \frac{c_b m_b - k_{ib} - k_{be}}{c_b m_b} x_2(j) + \frac{k_{be}}{c_b m_b} u_2(j) = f_2(x(t), u(t)) \end{cases}$$

$$\Downarrow$$

a system of first-order difference equations called **state equations** is then derived:

$$x(j+1) = f(x(j), u(j)) = \begin{bmatrix} f_1(x(j), u(j)) \\ f_2(x(j), u(j)) \end{bmatrix}$$

About the variable of interest (air temperature T_i), the **output equation** is:

$$y(j) = T_i(j\Delta t) = x_1(j) = \eta(x(j), u(j))$$

where $y(j)$: **output variable** of the system

- If $m \neq 1$, state equations are nonlinear (with respect to x and u)
- If all the coefficients are constant and $m = 1 \Rightarrow$
  a **linear time-invariant (LTI)** model is derived:

$$\begin{aligned} x(j+1) &= Ax(j) + Bu(j) \\ y(j) &= Cx(j) + Du(j) \end{aligned}$$

where A, B, C and D are suitable constant matrices:

$$A = \begin{bmatrix} \frac{c_i m_i - k_{ib} - k_{ic}}{c_i m_i} & \frac{k_{ib}}{c_i m_i} \\ \frac{k_{ib}}{c_b m_b} & \frac{c_b m_b - k_{ib} - k_{be}}{c_b m_b} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{k_{ic}}{c_i m_i} & 0 \\ 0 & \frac{k_{be}}{c_b m_b} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Input-output representations

- At first, linear dynamics systems will be considered (much simpler case); later, also nonlinear dynamic systems will be analyzed
- Some mathematical instruments are necessary:
  - the Laplace transform for continuous-time linear time-invariant systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \qquad \forall t \in \mathbb{R}$$

  - the z-transform for discrete-time linear time-invariant systems

$$\begin{aligned} x(j+1) &= Ax(j) + Bu(j) \\ y(j) &= Cx(j) + Du(j) \end{aligned} \qquad \forall j \in \mathbb{N}$$

- Input-output representation can be directly derived from state variable models

(Unilateral) Laplace transform

Laplace transform takes as argument a continuous-time real function $f(t) : \mathbb{R} \to \mathbb{R}$ and returns a complex valued function $F(s) : \mathbb{C} \to \mathbb{C}$ defined as:

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_{t=0_-}^{\infty} f(t)\, e^{-st} dt, \quad s = \sigma + j\omega \in \mathbb{C}$$

Main properties:

- The Laplace transform $\mathcal{L}$ is invertible, with a unique inverse transform $\mathcal{L}^{-1}$
- Both $\mathcal{L}$ and $\mathcal{L}^{-1}$ are linear operators: $\mathcal{L}\left[\alpha_1 f_1(t) + \alpha_2 f_2(t)\right] = \alpha_1 F_1(s) + \alpha_2 F_2(s)$
- The Laplace transforms of the time derivatives of $f(t)$ are:

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(t = 0_-)$$

$$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2} = \frac{d\frac{df(t)}{dt}}{dt} = \frac{dg(t)}{dt}\right] = sG(s) - g(t = 0_-) =$$

$$= s^2 F(s) - s \cdot f(t = 0_-) - \left.\frac{df(t)}{dt}\right|_{t=0_-}$$

- Laplace transforms can be applied to continuous-time LTI state representations, transforming linear differential equations into linear algebraic equations:

$$\begin{cases} \mathcal{L}\left[\dot{x}\left(t\right)\right] = sX\left(s\right) - x\left(t = 0_{-}\right) = sX\left(s\right) - x_0 \\ \mathcal{L}\left[Ax\left(t\right) + Bu\left(t\right)\right] = AX\left(s\right) + BU\left(s\right) \end{cases} \Rightarrow$$

$$sX\left(s\right) - x_0 = AX\left(s\right) + BU\left(s\right) \quad \Rightarrow$$

$$sX\left(s\right) - AX\left(s\right) = \left(sI - A\right)X\left(s\right) = x_0 + BU\left(s\right)$$

If $\det\left(sI - A\right) \neq 0$, then $\left(sI - A\right)$ is invertible $\Rightarrow \exists$ a unique $\left(sI - A\right)^{-1}$. The values of s that give $\det\left(sI - A\right) = 0$ are the eigenvalues of A, whose number is equal to the dimension n of the matrix A, because $\det\left(sI - A\right)$ is a polynomial of degree n in the variable s ($\Leftarrow$ fundamental theorem of algebra) $\Rightarrow$ with the exception of n values of s for which $\det\left(sI - A\right) = 0$:

$$X\left(s\right) = \left(sI - A\right)^{-1}x_0 + \left(sI - A\right)^{-1}BU\left(s\right)$$

- Moreover:

$$\begin{cases} \mathcal{L}\left[y\left(t\right)\right] = Y\left(s\right) \\ \mathcal{L}\left[Cx\left(t\right) + Du\left(t\right)\right] = CX\left(s\right) + DU\left(s\right) \end{cases} \Rightarrow$$

$$Y\left(s\right) = CX\left(s\right) + DU\left(s\right) \quad \Rightarrow$$

$$Y\left(s\right) = C\left(sI - A\right)^{-1} x_0 + \left[C\left(sI - A\right)^{-1} B + D\right] U\left(s\right)$$

- In particular, for $x_0 = 0$ (i.e., the system is initially at rest):

$$Y\left(s\right) = H\left(s\right) U\left(s\right)$$

$$H\left(s\right) = C\left(sI - A\right)^{-1} B + D$$: **transfer matrix** of the system $\in \mathbb{C}^{q \times p}$

since in general

$$u\left(t\right) = \begin{bmatrix} u_1\left(t\right) \\ \vdots \\ u_p\left(t\right) \end{bmatrix} \in \mathbb{R}^p, \quad y\left(t\right) = \begin{bmatrix} y_1\left(t\right) \\ \vdots \\ y_q\left(t\right) \end{bmatrix} \in \mathbb{R}^q$$

with $p =$ number of inputs, $q =$ number of outputs

- In the SISO case ($p = q = 1$), $H(s)$ is simply a **transfer function**, whose form is a rational function, i.e. a ratio of two polynomials in s:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{N_H(s)}{D_H(s)} =$$
$$= \frac{Ds^n + b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{\det(sI - A)} = \frac{Ds^n + b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n}$$

where the degree of the two polynomials is not greater than $n =$ system order $=$ number of states $=$ dimension of the state vector x

  - The denominator of $H(s)$ is the characteristic polynomial $p_c(s)$ of A, defined as $\det(sI - A)$, whose roots are the eigenvalues of A for which no solution exists for the equation $(sI - A) X(s) = x_0 + BU(s)$
  - Zeros z_i of $H(s)$: roots of numerator polynomial $N_H(s)$
  - Poles p_i of $H(s)$: roots of denominator polynomial $D_H(s)$
  - In general, $H(s)$ can be factorized as:

$$H(s) = D \frac{\prod_{i=1}^{n} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)} \text{ if } D \neq 0, \quad H(s) = b_1 \frac{\prod_{i=1}^{n-1} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)} \text{ if } D = 0$$

- In the MIMO case ($p > 1$ and/or $q > 1$), $H(s)$ is a matrix in $\mathbb{C}^{q \times p}$ whose elements are transfer functions:

$$H(s) = \begin{bmatrix} H_{11}(s) & \cdots & H_{1j}(s) & \cdots & H_{1p}(s) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ H_{i1}(s) & \cdots & H_{ij}(s) & \cdots & H_{ip}(s) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ H_{q1}(s) & \cdots & H_{qj}(s) & \cdots & H_{qp}(s) \end{bmatrix}$$

where $H_{ij}(s)$ is the transfer function between the j-th input $U_j(s)$ and the i-th output $Y_i(s)$, assuming that $U_k(s) = 0 \ \forall k \neq j$, since:

$$Y_i(s) = \sum_{j=1}^{p} H_{ij}(s) U_j(s) \Rightarrow$$

$$H_{ij}(s) = \left. \frac{Y_i(s)}{U_j(s)} \right|_{U_k(s)=0, \forall k \neq j}$$

(Unilateral) z-transform

z-transform takes as argument a discrete-time real function $f(k) : \mathbb{N} \rightarrow \mathbb{R}$ and returns a complex valued function $F(z) : \mathbb{C} \rightarrow \mathbb{C}$ defined as:

$$F(z) = \mathcal{Z}\left[f(k)\right] = \sum_{k=0}^{\infty} f(k)\, z^{-k}, \quad z = |z|\, e^{j\arg(z)} \in \mathbb{C}$$

Main properties:

- The z-transform $\mathcal{Z}$ is invertible, with a unique inverse transform $\mathcal{Z}^{-1}$
- Both $\mathcal{Z}$ and $\mathcal{Z}^{-1}$ are linear operators
- The z-transform of the one-step left (or forward or in advance) shifted function $f(k+1)$ is:
$$\mathcal{Z}\left[f(k+1)\right] = zF(z) - z \cdot f(k=0)$$
- The z-transform of the l-step right (or backward) shifted function $f(k-l)$ is:
$$\mathcal{Z}\left[f(k-l)\right] = z^{-l} F(z)$$

- The z-transform can be applied to discrete-time LTI state representations, transforming linear difference equations into linear algebraic equations:

$$\begin{cases} \mathcal{Z}\left[x\left(j+1\right)\right] = zX\left(z\right) - z \cdot x\left(j=0\right) = zX\left(z\right) - z \cdot x_0 \\ \mathcal{Z}\left[Ax\left(j\right)+Bu\left(j\right)\right] = AX\left(z\right) + BU\left(z\right) \end{cases} \Rightarrow$$

$$zX\left(z\right) - z \cdot x_0 = AX\left(z\right) + BU\left(z\right) \quad \Rightarrow$$

$$zX\left(z\right) - AX\left(z\right) = \left(zI-A\right)X\left(z\right) = z \cdot x_0 + BU\left(z\right)$$

If $\det\left(zI-A\right) \neq 0$, then $\left(zI-A\right)$ is invertible $\Rightarrow \exists$ a unique $\left(zI-A\right)^{-1}$. The values of z that give $\det\left(zI-A\right)=0$ are the eigenvalues of A, whose number is equal to the dimension n of the matrix A, because $\det\left(zI-A\right)$ is a polynomial of degree n in the variable z ($\Leftarrow$ fundamental theorem of algebra) $\Rightarrow$ with the exception of n values of s for which $\det\left(zI-A\right)=0$:

$$X\left(z\right) = z \cdot \left(zI-A\right)^{-1} x_0 + \left(zI-A\right)^{-1} BU\left(z\right)$$

- Moreover:

$$\begin{cases} \mathcal{Z}\left[y\left(j\right)\right] = Y\left(z\right) \\ \mathcal{Z}\left[Cx\left(j\right) + Du\left(j\right)\right] = CX\left(z\right) + DU\left(z\right) \end{cases} \Rightarrow$$

$$Y\left(z\right) = CX\left(z\right) + DU\left(z\right) \quad \Rightarrow$$

$$Y\left(z\right) = z \cdot C\left(zI - A\right)^{-1} x_0 + \left[C\left(zI - A\right)^{-1} B + D\right] U\left(z\right)$$

- In particular, for $x_0 = 0$ (i.e., the system is initially at rest):

$$Y\left(z\right) = H\left(z\right) U\left(z\right)$$

$$H\left(z\right) = C\left(zI - A\right)^{-1} B + D$$: **transfer matrix** of the system $\in \mathbb{C}^{q \times p}$

since in general

$$u\left(t\right) = \begin{bmatrix} u_1\left(t\right) \\ \vdots \\ u_p\left(t\right) \end{bmatrix} \in \mathbb{R}^p, \quad y\left(t\right) = \begin{bmatrix} y_1\left(t\right) \\ \vdots \\ y_q\left(t\right) \end{bmatrix} \in \mathbb{R}^q$$

with $p =$ number of inputs, $q =$ number of outputs

Input-output representation in regression form

The input-output representation of a SISO discrete-time LTI model, for $x_0 = 0$, is:

$$Y(z) = H(z)\,U(z) = \frac{Dz^n + b_1 z^{n-1} + b_2 z^{n-2} + \ldots + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n} U(z)$$

By taking some simple algebraic manipulation:

$$\left(z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n\right) \cdot Y(z) = \left(Dz^n + b_1 z^{n-1} + b_2 z^{n-2} + \ldots + b_n\right) \cdot U(z)$$

$$z^n Y(z) + a_1 z^{n-1} Y(z) + a_2 z^{n-2} Y(z) + \ldots + a_n Y(z) =$$
$$= Dz^n U(z) + b_1 z^{n-1} U(z) + b_2 z^{n-2} U(z) + \ldots + b_n U(z)$$

dividing by z^n and then leaving only the term $Y(z)$ in the left hand side:

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \ldots + a_n z^{-n} Y(z) =$$
$$= DU(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \ldots + b_n z^{-n} U(z)$$

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \ldots - a_n z^{-n} Y(z) +$$
$$+ DU(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \ldots + b_n z^{-n} U(z)$$

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \ldots - a_n z^{-n} Y(z) + \\ + DU(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \ldots + b_n z^{-n} U(z)$$

$$\Downarrow \mathcal{Z}^{-1}$$

$$\begin{aligned} y(j) &= -a_1 y(j-1) - a_2 y(j-2) - \ldots - a_n y(j-n) + \\ &\qquad + Du(j) + b_1 u(j-1) + b_2 u(j-2) + \ldots + b_n u(j-n) \\ &= -\sum_{i=1}^{n} a_i y(j-i) + Du(j) + \sum_{i=1}^{n} b_i u(j-i) \end{aligned}$$

(input-output representation in **regression form** for a SISO discrete-time LTI model)

- For a LTI model of dimension n, the output $y(j)$ is a linear combination of:
  - the n past values of the output from $y(j-1)$ up to $y(j-n)$
  - the n past values of the input from $u(j-1)$ up to $u(j-n)$
  - possibly the input $u(j)$, if $D \neq 0$

  Neither output values from $y(j-n-1)$ to $y(j-\infty)$, neither past input values from $u(j-n-1)$ to $u(j-\infty)$ have influence on $y(j)$
- **Problem**: estimate $a_1, a_2, \ldots, a_n, D, b_1, b_2, \ldots, b_n$ from experimental data

A finite number N of measurements of input $u(\cdot)$ and output $y(\cdot)$ is performed, supposing for simplicity that there is not measurement error $\Rightarrow \forall j \in [n+1, N]$:

$$y(j) = -a_1 y(j-1) - \ldots - a_n y(j-n) + Du(j) + b_1 u(j-1) + \ldots + b_n u(j-n)$$

$$\Downarrow$$

a set of $N-n$ linear algebraic equations is obtained:

$$\begin{cases} y(n+1) = -a_1 y(n) - \ldots - a_n y(1) + Du(n+1) + b_1 u(n) + \ldots + b_n u(1) \\ y(n+2) = -a_1 y(n+1) - \ldots - a_n y(2) + Du(n+2) + b_1 u(n+1) + \ldots + b_n u(2) \\ \quad\vdots \quad = \quad\quad \vdots \\ y(N) = -a_1 y(N-1) - \ldots - a_n y(N-n) + Du(N) + b_1 u(N-1) + \ldots + b_n u(N-n) \end{cases}$$

$$\underbrace{\begin{bmatrix} y(n+1) \\ y(n+2) \\ \vdots \\ y(N) \end{bmatrix}}_{y^N \in \mathbb{R}^{N-n}} = \underbrace{\begin{bmatrix} -y(n) & \cdots & -y(1) & u(n+1) & u(n) & \cdots & u(1) \\ -y(n+1) & \cdots & -y(2) & u(n+2) & u(n+1) & \cdots & u(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -y(N-1) & \cdots & -y(N-n) & u(N) & u(N-1) & \cdots & u(N-n) \end{bmatrix}}_{L \in \mathbb{R}^{(N-n)\times(2n+1)}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ D \\ b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\theta \in \mathbb{R}^{2n+1}}$$

$$\underbrace{\begin{bmatrix} y(n+1) \\ y(n+2) \\ \vdots \\ y(N) \end{bmatrix}}_{y^N \in \mathbb{R}^{N-n}} = \underbrace{\begin{bmatrix} -y(n) & \cdots & -y(1) & u(n+1) & u(n) & \cdots & u(1) \\ -y(n+1) & \cdots & -y(2) & u(n+2) & u(n+1) & \cdots & u(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -y(N-1) & \cdots & -y(N-n) & u(N) & u(N-1) & \cdots & u(N-n) \end{bmatrix}}_{L \in \mathbb{R}^{(N-n)\times(2n+1)}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ D \\ b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\theta \in \mathbb{R}^{2n+1}}$$

$$\Downarrow$$

$$y^N = L \cdot \theta$$

- y^N, L : known matrices (depend only on measurements $y(\cdot)$ and $u(\cdot)$)
- θ : unknown vector

- **Estimation problem**: how to evaluate θ from experimental data?
  - If L is square (i.e., $N = 3n + 1$) and invertible $\Rightarrow \theta = L^{-1} \cdot y^N$
  - Otherwise?