01RKYQW - Estimation, filtering, and system identification
Report of the sample examination paper

Problem #1. The mean values are removed from the input-output measurements, to obtain zero mean value sequences of length $N_{tot}$. Then the overall experimental data are partitioned in two datasets: one of length $N_c$ for model identification, the other of length $N_v$ for model validation. For example, the same length may be chosen for each dataset, i.e., $N_c = N_v = N_{tot}/2 = 1000$.

1) The estimation dataset is used to identify ARX, ARMAX and OE models of different orders and delays, and to look for models that guarantee satisfactory characteristics of whiteness of the residuals:

- ARX models of order $n_a = n_b$ and delay $n_k \in [1, 2, 3]$ require $n_a \geq 4$ to have few (no more than 3) values of the autocorrelation function of residuals outside enough the 99% confidence limits;
- ARMAX models of order $n_a = n_b = n_c$ and delay $n_k \in [1, 2, 3]$ require $n_a \geq 2$ to have few (no more than 3) values of the autocorrelation function of residuals outside enough the 99% confidence limits;
- OE models of order $n_f = n_a$ and delay $n_k = 1$ or $n_k \in [2, 3]$ require $n_f \geq 3$ or $n_f \in [3, 4, 5, 7]$, respectively, to have few (no more than 3) values of the autocorrelation function of residuals outside enough the 99% confidence limits.

2) The validation dataset is then used to compare the identified models and to assess their model quality, by minimizing the Root Mean Square Error $\text{RMSE} = \sqrt{\frac{1}{N_o - N_0 + \sum_{k=N_0+1}^{N_o} (y(t) - \hat{y}(t))^2}}$, where $y(t)$ = measured output, $\hat{y}(t)$ = simulated (or predicted) output and $N_0 = 10$ is chosen as suitable time instant after which the transient is past. Note that, in case of ARX and ARMAX models, the predicted output provides better performance than the simulated output, since it exploits more information. For this reason, the values of the RMSE using the predicted output only are here reported for all the models:

<table>
<thead>
<tr>
<th>Identified model</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
<th>$k = 5$</th>
<th>$k = 6$</th>
<th>$k = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARX($n_a = k$, $n_b = k$, $n_k = 1$)</td>
<td>0.0691</td>
<td>0.0664</td>
<td>0.0670</td>
<td>0.0660</td>
<td>0.0638</td>
<td>0.0617</td>
<td>0.0603</td>
</tr>
<tr>
<td>ARX($n_a = k$, $n_b = k$, $n_k = 2$)</td>
<td>0.0688</td>
<td>0.0647</td>
<td>0.0668</td>
<td><strong>0.0656</strong></td>
<td>0.0638</td>
<td>0.0617</td>
<td>0.0603</td>
</tr>
<tr>
<td>ARX($n_a = k$, $n_b = k$, $n_k = 3$)</td>
<td>0.0680</td>
<td>0.0653</td>
<td>0.0662</td>
<td>0.0656</td>
<td>0.0639</td>
<td>0.0618</td>
<td>0.0605</td>
</tr>
<tr>
<td>ARMAX($n_a = n_b = n_c = k$, $n_k = 1$)</td>
<td>0.0698</td>
<td>0.0540</td>
<td>0.0560</td>
<td>0.0504</td>
<td>0.0500</td>
<td>0.0521</td>
<td>0.0510</td>
</tr>
<tr>
<td>ARMAX($n_a = n_b = n_c = k$, $n_k = 2$)</td>
<td>0.0697</td>
<td>0.0534</td>
<td>0.0536</td>
<td>0.0502</td>
<td>0.0552</td>
<td>0.0508</td>
<td>0.0524</td>
</tr>
<tr>
<td>ARMAX($n_a = n_b = n_c = k$, $n_k = 3$)</td>
<td>0.0681</td>
<td><strong>0.0532</strong></td>
<td>0.0510</td>
<td>0.0508</td>
<td>0.0510</td>
<td>0.0549</td>
<td>0.0519</td>
</tr>
<tr>
<td>OE($n_a = k$, $n_f = k$, $n_k = 1$)</td>
<td>0.1234</td>
<td>0.0584</td>
<td><strong>0.0504</strong></td>
<td>0.0507</td>
<td>0.0499</td>
<td>0.0507</td>
<td>0.0511</td>
</tr>
<tr>
<td>OE($n_a = k$, $n_f = k$, $n_k = 2$)</td>
<td>0.1167</td>
<td>0.0976</td>
<td>0.0505</td>
<td>0.0506</td>
<td>0.0509</td>
<td>0.0786</td>
<td>0.0507</td>
</tr>
<tr>
<td>OE($n_a = k$, $n_f = k$, $n_k = 3$)</td>
<td>0.1115</td>
<td>0.0538</td>
<td>0.0514</td>
<td>0.0511</td>
<td>0.0511</td>
<td>0.0737</td>
<td>0.0516</td>
</tr>
</tbody>
</table>

For ARX models, the best trade-off between RMSE and model complexity $n = n_a + n_b$ that guarantees satisfactory characteristics of whiteness of the residuals is given by the ARX($4, 4, 2$), with $\text{RMSE} = 0.0656$ and $n = 8$. For ARMAX models, the best trade-off between RMSE and model complexity $n = n_a + n_b + n_c$ that guarantees satisfactory characteristics of whiteness of the residuals is the ARMAX($2, 2, 2, 3$), with $\text{RMSE} = 0.0532$ and $n = 6$. For OE models, the best trade-off between RMSE and model complexity $n = n_a + n_f$ that guarantees satisfactory characteristics of whiteness of the residuals is given by the OE($3, 3, 1$), with $\text{RMSE} = 0.0504$ and $n = 6$. In summary, the best trade-off between RMSE and model complexity $n$ that at the same time guarantees satisfactory characteristics of whiteness of the residuals is provided by the OE($3, 3, 1$).

3) Using all the experimental data, the following parameters of an ARX($3, 3, 1$) model have been estimated by means of the standard Least-Squares algorithm:

\[
\hat{\theta}_1 = \hat{a}_1 = -0.6566, \quad \hat{\theta}_2 = \hat{a}_2 = -0.3219, \quad \hat{\theta}_3 = \hat{a}_3 = 0.1074, \quad \hat{\theta}_4 = \hat{b}_1 = 0.0195, \quad \hat{\theta}_5 = \hat{b}_2 = 0.0328, \quad \hat{\theta}_6 = \hat{b}_3 = 0.0832
\]

Assuming that the output measurements are corrupted by an energy-bounded noise whose 2-norm is less than 4, the following Estimate Uncertainty Intervals $\text{EUI}^2$ are derived:

\[
\hat{\theta}_1 = \hat{a}_1 \in [-1.9391, 0.6260]; \quad \hat{\theta}_2 = \hat{a}_2 \in [-1.8080, 1.1641]; \quad \hat{\theta}_3 = \hat{a}_3 \in [-1.0511, 1.2658]; \\
\hat{\theta}_4 = \hat{b}_1 \in [-0.7246, 0.7636]; \quad \hat{\theta}_5 = \hat{b}_2 \in [-0.9347, 1.0004]; \quad \hat{\theta}_6 = \hat{b}_3 \in [-0.6985, 0.8650]
\]

Since the fitting error $||y - \Phi \hat{\theta}|| = 3.05$ is less than the 2-norm noise bound, the Parameter Uncertainty Intervals $\text{PUI}^2$ are given by:

\[
\hat{\theta}_1 = \hat{a}_1 \in [-1.4853, 0.1721]; \quad \hat{\theta}_2 = \hat{a}_2 \in [-1.2821, 0.6382]; \quad \hat{\theta}_3 = \hat{a}_3 \in [-0.6411, 0.8558]; \\
\hat{\theta}_4 = \hat{b}_1 \in [-0.4613, 0.5003]; \quad \hat{\theta}_5 = \hat{b}_2 \in [-0.5924, 0.6580]; \quad \hat{\theta}_6 = \hat{b}_3 \in [-0.4219, 0.5884]
\]

Assuming that the output measurements are corrupted by an amplitude-bounded noise whose $\infty$-norm is less than 0.1, the following Estimate Uncertainty Intervals $\text{EUI}^\infty$ are derived:

\[
\hat{\theta}_1 = \hat{a}_1 \in [-1.7464, 0.4332]; \quad \hat{\theta}_2 = \hat{a}_2 \in [-1.6435, 0.9996]; \quad \hat{\theta}_3 = \hat{a}_3 \in [-0.8615, 1.0762]; \\
\hat{\theta}_4 = \hat{b}_1 \in [-0.3781, 0.4171]; \quad \hat{\theta}_5 = \hat{b}_2 \in [-0.5767, 0.6423]; \quad \hat{\theta}_6 = \hat{b}_3 \in [-0.3737, 0.5402]
\]
Problem #2. First the steady-state Kalman filter $F_\infty$ and the dynamic Kalman 1-step predictor in predictor-corrector form $K_{pc}$ are designed, assuming $x(1) = [50, 100, 50]$ as initial state of the system $S_2$.

2) The state estimates provided by $F_\infty$ and $K_{pc}$ have been compared by evaluating the Root Mean Square Errors:

$$RMSE_k = \sqrt{\frac{1}{N' - N'_0} \sum_{t=N'_0+1}^{N'} [x_k(t) - \hat{x}_k(t)]^2}, \quad k = 1, \ldots, 3$$

where $\hat{x}_k(t)$ is the estimate of the state $x_k(t)$ and $N'_0 = 100$ is chosen as suitable time instant after which the filter or predictor transient is past.

Note that the values of $RMSE$ depend on the realizations of the white noises $v_1(t)$ and $v_2(t)$. For example:

- using the dynamic Kalman 1-step predictor $K_{pc}$, $RMSE = \begin{bmatrix} 0.0568 \\ 0.1114 \\ 0.0552 \end{bmatrix}$;

- using the steady-state Kalman filter $F_\infty$, $RMSE = \begin{bmatrix} 0.0561 \\ 0.1113 \\ 0.0556 \end{bmatrix}$.

Note moreover that, if $N'_0 = 0$ is chosen, then:

- using the dynamic Kalman 1-step predictor $K_{pc}$, $RMSE = \begin{bmatrix} 3.4358 \\ 7.2297 \\ 3.7862 \end{bmatrix}$;

- using the steady-state Kalman filter $F_\infty$, $RMSE = \begin{bmatrix} 3.6312 \\ 7.5607 \\ 3.9285 \end{bmatrix}$.